DOUBLE BINARY FORMS WITH THE CLOSURE PROPERTY*

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INTRODUCTION

The polygons of Poncelet—a variable polygon whose n vertices run over a fixed conic C while its n sides touch a fixed conic K—have long excited the interest of geometers.† If t is the parameter of a variable point of C, τ the parameter of a variable tangent of K, the incidence condition of point t of C and line τ of K is a double $(2,2)$ binary form

$$(\alpha t)^2(a\tau)^2=0 \tag{1}$$

since each point t of C is on two lines τ of K and each line τ of K is on two points t of C. When the two conics C, K are so situated with respect to each other that the variable Poncelet polygons can be drawn, the double form (1) has the *closure property*. For a given vertex t_1 of a given polygon P_n of n sides determines in (1) the two sides τ_1, τ_n on t_1. On each of these sides there is a vertex t_2, t_n respectively in addition to t_1, the additional vertices being determined from (1). From these vertices additional sides τ_2, τ_{n-1} are obtained, etc., until finally the polygon closes in to form a *configuration* $\Delta_{n,n}^{2,2}$ with parameters $t_1, \cdots, t_n; \tau_1, \cdots, \tau_n$ of such sort that in (1) each of the n t's determines two of the n τ's and vice versa. We shall say then that the form (1) *admits* the configuration $\Delta_{n,n}^{2,2}$. If the form admits ∞^1 such configurations, i. e., if it has the closure property for any initial t_1 (or τ_1) then it is said to be *poristic*. A beautiful theorem‡ of Poncelet states that the existence of one polygon P_n implies the existence of ∞^1 polygons. Thus if the form (1) admits *one* $\Delta_{n,n}^{2,2}$ it admits infinitely many and is poristic.

A new departure in this field is due to H. S. White§ who showed that there exist $(3,3)$ double forms

$$(\alpha t)^3(a\tau)^3 \tag{2}$$

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† Cf. G. Loria, *I polygoni di Poncelet*, Turin, 1889; H. S. White, *Poncelet polygons*, Science, Feb. 4, 1916, pp. 149–58.

‡ Poncelet, *Propriétés Projectives des Figures*, 1822, p. 565 ff.

§ Proceedings of the National Academy of Sciences, vol. 1 (1915), p. 464; vol. 2 (1916), p. 337.

with configurations $\Delta_{7,7}^{3,3}$. Here again the existence of one configuration implies the existence of ∞^1 configurations. If, in space, (2) is the incidence condition of point t of a twisted cubic C^3 with plane τ of a twisted cubic K^3 the configuration $\Delta_{7,7}^{3,3}$ consists of 7 points of C^3 and 7 planes of K^3 such that each of the 7 points is on three of the 7 planes and vice versa. It is however necessary in this case to specify the arrangement of incidences in the configuration. This may be done conveniently by giving the seven triads of points $t_1, \cdots, t_7$ (or merely their subscripts $1, \cdots, 7$) cut out by the seven planes $\tau_1, \cdots, \tau_7$ as in the scheme

$$127 \,.\, 136 \,.\, 145 \,.\, 235 \,.\, 246 \,.\, 347 \,.\, 567 \,.$$

A few years later the writer* discussed the conditions for closure of the general double (k, κ) binary form

$$F = (\alpha t)^k (a\tau)^\kappa = 0 \,. \tag{3}$$

If there exists a set of n values $t_1, \cdots, t_n$ of t and a set of ν values $\tau_1, \cdots, \tau_\nu$ of τ such that each of the n t's determines in (3) κ of the ν τ's and each of the ν τ's determines in (3) k of the n t's where

$$n\kappa = \nu k \,, \tag{4}$$

then F is said to admit the configuration $\Delta_{n,\nu}^{k,\kappa}$. If F admits ∞^1 such configurations the double form has the closure property and is poristic. For such poristic forms certain fundamental theorems were derived. Thus the sets of n values t lie in an involution $(\gamma t)^n + \lambda(\delta t)^n = 0$; the sets of ν values τ lie in an involution $(c\tau)^\nu + \lambda(d\tau)^\nu = 0$ whose sets are projectively related to the sets of the first involution. Hence the determinant

$$D_{n,\nu} \equiv \begin{vmatrix} (\gamma t)^n & (\delta t)^n \\ (c\tau)^\nu & (d\tau)^\nu \end{vmatrix} = F \cdot F' \tag{5}$$

factors into the form F and a *complementary form*

$$F' = (\beta t)^{n-k} (b\tau)^{\nu-\kappa} \tag{6}$$

such that the form F' admits the configurations $\Delta_{n,\nu}^{n-k,\nu-\kappa}$ *complementary* to $\Delta_{n,\nu}^{k,\kappa}$, i. e., t in (6) determines the $\nu - \kappa$ values τ which are *not* determined in (3) by t. Conversely the determinant, $D_{n,\nu} = 0$, is obviously poristic with configurations $\Delta_{n,\nu}^{n,\nu}$ and, if factorable, its factors are necessarily poristic. Particularly notable is the theorem†

* A. B. Coble, *Multiple binary forms with the closure property*, American Journal of Mathematics, vol. 43 (1921), pp. 1–19.

† A. B. Coble, loc. cit., p. 3.

(7) *The necessary and sufficient conditions that a form* $F_{k,\kappa}$ *shall be poristic with configurations* $\Delta^{k,\kappa}_{n,\nu}(n,\nu,k,\kappa$ *integers subject to* (4)) *are* (A) *that the given form shall admit one such configuration and* (B) *that there shall exist a form* $F_{n-k,\nu-\kappa}$ *which shall admit the complementary configuration* $\Delta^{n-k,\nu-\kappa}_{n,\nu}$.

In addition to such theorems certain general methods of procedure are explained whose efficacy is shown by the derivation of twelve new types of poristic forms with configurations

$$\Delta^{n-2,n-2}_{n,n}\ ,\quad \Delta^{k,\kappa}_{k,\kappa}\ ,\quad \Delta^{k,k}_{k+1,k+1}\ ,\quad \Delta^{2,n}_{2n,n_2}\ ,\quad \Delta^{2,n-1}_{2n,n(n-1)}\ ,\quad \Delta^{2(n-1),2(n-1)}_{2n,n_2}\ ,$$

(8)

$$\Delta^{2(n-1),(n-1)^2}_{2n,n(n-1)}\ ,\quad \Delta^{4,4}_{7,7}\ ,\quad \Delta^{2,3}_{4,6}\ ,\quad \Delta^{2,4}_{5,10}\ ,\quad \Delta^{3,1}_{5,10}\ ,\quad \Delta^{3,4}_{6,8}\ ,$$

to which should be added the Poncelet type $\Delta^{2,2}_{n,n}$ and White's type $\Delta^{3,3}_{7,7}$. Six of these new types embody an infinite number of cases for arbitrary choice of the integer n. One type embodies a doubly infinite number of cases for arbitrary choice of k,κ. This type however is the obvious poristic determinant $D_{k,\kappa}$ in (5). The other types are particular cases selected because they were manageable.

Quite recently* Louise D. Cummings exhibited two poristic forms $F_{3,3}$ each with configurations $\Delta^{3,3}_{6,6}$ which she calls "double sextettes," the grouping for the two forms being essentially different. Professor Cummings' method is novel. She examines first the conditions on the coefficients of the form $F_{3,3}$ that it may admit *one* double sextette $\Delta^{3,3}_{6,6}$ of the prescribed kind. The additional integral conditions on the coefficients of the form that it may admit a consecutive sextette are then determined. If both types of conditions can be satisfied the resulting form is necessarily poristic.†

It occurred to the present author to examine these two porisms by the methods of his earlier paper.‡ This discussion is carried out in § 2 and § 4 after a preliminary discussion in § 1 of the associated group problem. A striking result of the geometric argument employed is that Miss Cummings'

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* Louise D. Cummings, *Proceedings International Mathematical Congress*, Toronto (1924), not yet published; Bulletin of the American Mathematical Society, vol. 31 (1925), pp. 266–74.

† These remarks are based on the exposition contained in the Bulletin paper. I have seen the manuscript of the Toronto paper but recall no details.

‡ Loc. cit.

poristic (3,3) form is *factorable** in each of the two cases—a fact not recognized by Miss Cummings and indeed not easily recognizable in the explicitly given form F.† Moreover, the character of the factors is so clearly defined that each case appears as an individual in a doubly infinite series. One of these series may properly be called the "Poncelet polygons of n sides in space of r dimensions." The other series has a characteristic grouping which, for reasons explained later, may be called "cyclically imprimitive." For each series the explicit algebraic construction of the poristic forms is given.

In § 6 all poristic forms $F_{3,3}$ with configurations $\Delta^{3,3}_{n,n}$ $(n<10)$ are determined. Of these all but two, the double sextettes of White, and a novel case of double octettes, are included in the general classes mentioned above. Except for these two and one other, all the poristic $F_{3,3}$'s $(n<10)$ are factorable.

1. The group problem for poristic forms

It has been pointed out‡ that a configuration $\Delta^{k,\kappa}_{n,\nu}$ is not defined until the n t's have been arranged in ν sets of k in such a way that each t is included in κ of the ν sets. A convenient way of indicating such an arrangement is to write down the k-ads formed from the t's and to assign to them in some order the values of the $\nu\tau$'s. Such an arrangement is given in the introduction for White's porism. However, not every such arrangement can lead to a porism. An additional condition is given in the following theorem:

(9) *For a poristic form $F_{k,\kappa}$ with configurations $\Delta^{k,\kappa}_{n,\nu}$ the arrangement of n t's into ν sets of k each must be such as will admit the permutations of a group transitive on the n t's.*

For if the form $F_{k,\kappa}$ $(k\leqq\kappa)$ is interpreted, as in the author's earlier paper,§ as the incidence condition of point, t, of a rational norm curve N^k in S_k with S_{k-1}, τ, of a rational curve R^κ in S_k, the points t of the ∞^1 configurations run over N^k and a configuration may be drawn starting with any one point t of a configuration in precisely the same way as by starting with any other point t' of the same configuration. Thus if t be replaced by t' it must be possible to replace any other t, say t_1, by some t, say t_1', in such a way that the configuration, i. e., the distribution of n t's into k-ads, is unaltered.

* I have a vague recollection of examining the configurations in connection with the preparation of my earlier paper (loc. cit.) and discarding them for some reason—probably this very factorization. A valuable feature of Miss Cummings' contribution is that examples are given of poristic forms which have considerable geometric interest in spite of their rather simple algebraic character.

† Bulletin of the American Mathematical Society, loc. cit., p. 268.

‡ A. B. Coble, loc. cit., p. 6.

§ Loc. cit., p. 7.

If we examine possible arrangements for the double sextettes $\Delta^{3,3}_{6,6}$ we find four cases:

(α) 123. 234. 345. 456. 561. 612;

(β) 142. 145. 253. 256. 361. 364;

(γ) 123. 124. 125. 456. 536. 346;

(δ) 123. 124. 135. 256. 346. 456.

We see that in (γ) the pair 12 occurs in three triads, the pairs 36, 46, 56 occur in two triads, and other pairs occur in one or no triads. Thus the group of (γ) is intransitive with 12, 345, 6 as systems of transitivity. Similarly in the triads of (δ) the pairs 12,13,46,56 each occur twice and all other pairs occur once except 16 which does not occur at all. Thus the group of (δ) is intransitive with 16,2345 as systems of transitivity. On the other hand (α) is invariant under a transitive dihedral $g_{2.6}$ and (β) is invariant under a transitive g_{24}. Both groups contain a cyclic element of period six as is noted by Miss Cummings. For (α) the cyclic g_6 is invariant under $g_{2.6}$; for (β), however, the cyclic g_6 is one of a number conjugate under g_{24}. Thus the emphasis is to be laid on the transitivity rather than the cyclic character of the group. It is, however, quite proper to call the configuration (α) *cyclic* or *polygonal*. We may better call the configuration (β) *cyclically imprimitive*. For in (β) the six t's divide into three cyclic pairs 14,25,36 and the triads are formed by combining a pair with a member of the next pair. For both (α) and (β) the groups are imprimitive with 14,25,36 as systems of imprimitivity.

An obvious further condition on the ν k-ads is that they shall not contain a set of $\nu_1<\nu$ k-ads such that in the ν_1 k-ads alone each t of a set of $n_1<n$ t's is included in κ of the ν_1 sets. In such case the given configuration would be merely an aggregate of smaller configurations.

In the next paragraph the case (β) is examined and the algebraic nature of its poristic form is determined.

2. The poristic form for the double sextette (β)

We will assume the existence of a poristic form $F_{3,3}$ with ∞^1 configurations $\Delta^{3,3}_{6,6}$ determined by the arrangement

(β) 142. 145. 253. 256. 361. 364,

and will call the triads of six t's in the order given $\tau_2,\tau_5,\tau_3,\tau_6,\tau_1,\tau_4$ respectively. Let then $G_{6.6}$ be that rational covariant of $F_{3,3}$ which expresses that t,t' determine in $F_{3,3}=0$ a common τ. This form $G_{6.6}$, symmetric in t,t',

coördinates to $t=t_1$ the values $t'=t_2,t_5,t_3,t_6$ and $t'=t_4$ taken twice. Hence $G_{6,6}$ is rationally factorable,

$$G_{6,6}=(I_{1,1})^2 \cdot K_{4,4} ,$$

where $I_{1,1}$ is a symmetric linear form in t,t', and $K_{4,4}$ a symmetric quartic form in t and in t'. Thus $I_{1,1}$ is an ordinary involution of pairs of points and t_1,t_4; t_2,t_5; t_3,t_6 are pairs of this involution. Similarly if $\bar{G}_{6,6}$ is the rational covariant of $F_{3,3}$ which expresses that τ,τ' determine in $F=0$ a common t, then $\bar{G}_{6,6}$ factors, i. e.,

$$\bar{G}_{6,6}=(\bar{I}_{1,1})^2 \cdot \bar{K}_{4,4}$$

where $\bar{I}_{1,1}$ is an involution with pairs τ_1,τ_4; τ_2,τ_5; $\tau_3\tau_6$.

The involutions $I_{1,1}$ and $\bar{I}_{1,1}$ can be obtained rationally from the forms G and therefore also from $F_{3,3}$, and on multiplying them respectively by $t-t'$, $\tau-\tau'$ the pairs of the involutions are obtained. We observe at once that the three τ's determined by a given t in $F_{3,3}=0$ divide into a pair of the involution $\bar{I}_{1,1}$ and an isolated τ; thus each t determines one τ and conversely each τ is thereby determined by one t. Hence the six t's and six τ's are projective and $F_{3,3}$ factors into

$$F_{3,3}=F_{1,1} \cdot F_{2,2} ,$$

where $F_{1,1}$ determines the projectivity between t and τ. For example t_1 determines in $F_{3,3}=0$ the pair τ_2,τ_5 of $\bar{I}_{1,1}$ and also τ_1; whence $F_{1,1}=0$ is a projectivity between t_i,τ_i $(i=1,\cdots,6)$.

The factors of $F_{3,3}$ are gotten by rational processes as follows. The resultant of $F_{3,3}$ and $\bar{I}_{1,1}$ in τ is of degree 3 in t and 3 in τ' and on replacing τ' by τ this resultant coördinates to t_1 the values $\tau=\tau_2,\tau_5,\tau_4$, whereas $F_{3,3}$ itself coördinates to t_1 the values $\tau=\tau_2,\tau_5,\tau_1$. Thus $F_{2,2}$ is the common factor of $F_{3,3}$ and this resultant, and $F_{1,1}$ is the residual factor of $F_{3,3}$.

For a canonical form we naturally take the projectivity $F_{1,1}=0$ to be the identical projectivity $t-\tau=0$ and thus regard t,τ as superposed ranges. If we take $0,\infty$ to be the double points of $I_{1,1}$ and therefore also of $\bar{I}_{1,1}$ the pairs are respectively $t,-t$ and $\tau,-\tau$. Moreover, the factor $F_{2,2}=0$ coördinates either to t or its partner $-t$ the same pair $\tau,-\tau$, whence $F_{2,2}=0$ has the form

$$a\tau^2t^2+b\tau^2+ct^2+d=0 .$$

Thus the form is *imprimitive** and arises from $a\sigma s+b\sigma+cs+d=0$ by the substitutions $\sigma=\tau^2$, $s=t^2$. This projectivity between σ and s (when each

* A. B. Coble, loc. cit., p. 6.

is on the same scale) is periodic of period three. For $t, -t=t_1,t_4$ determine, in $F_{2,2}=0, \tau, -\tau=\tau_2,\tau_5$ which in the projectivity $\tau=t$ (or $s=\sigma$) are $t, -t=t_2,t_5$. Similarly we pass from the pair t_2,t_5 to the pair t_3,t_6 and again to the original pair t_1,t_4. If the fixed points of the projectivity between s and σ are taken to be $1, f$ then the final canonical form of the required poristic form $F_{3,3}$ is

$$F_{3,3}=(t-\tau)\left\{\frac{t^2-f}{t^2-1}-\epsilon\,\frac{\tau^2-f}{\tau^2-1}\right\} \qquad (\epsilon=e^{2\pi i/3}) \,. \tag{10}$$

Thus $F_{3,3}$ has one absolute constant.

We observe that each factor of $F_{3,3}$ is poristic in a rather trivial way. Thus $t-\tau$ admits configurations $\Delta^{1,1}_{1,1}$ and the residual factor admits configurations $\Delta^{2,2}_{2,2}$ of *imprimitive* type. The effect of taking the product is to distribute the $\Delta^{2,2}_{2,2}$ into closed systems of three which form the double sextettes.

Miss Cummings gives the following as a normal form for this case:

$$\begin{aligned} F(x,y)= \quad & 12.35.56\,.\ ae.cf\,.\ \ xb.xc.xd\,.\ \ y1.y2.y4 \\ & +61.25.14\,.\ bf.ce\,.\ \ xa.xc.xd\,.\ \ y2.y3.y5 \quad (xa=x-a) \\ & +12.15.25\,.\ ce.cf\,.\ \ xa.xb.xd\,.\ \ y3.y4.y6 \quad (12=t_1-t_2) \\ & +15.23.26\,.\ cf.de\,.\ \ xa.xb.xc\,.\ \ y4.y5.y1 = 0 \end{aligned}$$

where a,b,c,d,e,f correspond to $\tau_2,\tau_3,\tau_4,\tau_5,\tau_6,\tau_1$ above. She finds as the condition for a single $\Delta^{3,3}_{6,6}$ the projectivity of the six t's and six τ's and as the additional condition for a porism the fact that $t_1,t_4;\ t_2,t_5;\ t_3,t_6$ are pairs of an involution. If then in $F(x,y)$ the six t's and six τ's in order are replaced by $\lambda,\mu,\nu,-\lambda,-\mu,-\nu$ the form $F(x,y)$ becomes, after a rather awkward reduction,

$$\begin{aligned} 4\lambda\mu(x-y)\,[\,&(\lambda^4+\mu^4+\nu^4-\mu^2\nu^2-\nu^2\lambda^2-\lambda^2\mu^2)x^2y^2 \\ &+(3\lambda^2\mu^2\nu^2-\lambda^4\nu^2-\mu^4\lambda^2-\nu^4\mu^2)x^2+(3\lambda^2\mu^2\nu^2-\lambda^4\mu^2-\mu^4\nu^2-\nu^4\lambda^2)y^2 \\ &+(\mu^4\nu^4+\nu^4\lambda^4+\lambda^4\mu^4-\{\lambda^2+\mu^2+\nu^2\}\lambda^2\mu^2\nu^2)\,] \,. \end{aligned}$$

The second factor is satisfied by $x^2,y^2=\lambda^2,\nu^2$ and therefore, due to the invariance of the coefficients under cyclic permutation of λ^2,μ^2,ν^2, is satisfied also by $x^2,y^2=\mu^2,\lambda^2$ and ν^2,μ^2, whence the period 3 and normal form (10) of the second factor is confirmed.

3. The poristic form $F_{r+1,r+1}$ with cyclically imprimitive configurations $\Delta^{r+1,r+1}_{rs,rs}$ and related poristic forms

Consider an array of rs values of t arranged in matrix form with s rows of r elements each as in

$$\begin{matrix} t_{1,1}\, t_{1,2} \cdot \cdot \cdot t_{1,r} \\ t_{2,1}\, t_{2,2} \cdot \cdot \cdot t_{2,r} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ t_{s,1}\, t_{s,2} \cdot \cdot \cdot t_{s,r}. \end{matrix}$$

We consider the rs sets of $r+1$ t's obtained by adjoining to the set of r t's in one row any one t of the next lower row and to the set of r t's in the last row any one t of the first row. The typical set is then

$$\tau_{i+1,j}: \quad t_{i,1}, t_{i,2}, \cdot \cdot \cdot, t_{i,r}, t_{i+1,j} \qquad \left(\begin{matrix} i=1, \cdot \cdot \cdot, s\,; s+1=1 \\ j=1, \cdot \cdot \cdot, r \end{matrix} \right).$$

We assume the existence of a poristic form $F_{r+1,r+1}$ with closed sets of rs values of t and rs values of τ such that the values of τ determine sets of $r+1$ t's as in the arrangement indicated. The set of τ's thereby associated to a given t is

$$t_{i+1,j}: \quad \tau_{i+2,1}, \tau_{i+2,2}, \cdot \cdot \cdot, \tau_{i+2,r}, \tau_{i+1,j}.$$

The form $G_{r(r+1),r(r+1)}$ symmetric in t and t', which expresses that t and t' determine in $F_{r+1,r+1}$ a common τ, coördinates to $t=t_{i,j}$ the values t' found in the row t_{i-1}, the row t_{i+1}, and also the values in the row t_i other than $t_{i,j}$ each taken r times. This r-fold factor can be isolated rationally and gives rise to a form $H_{r-1,r-1}$ symmetric in t,t' which for any t determines the remaining t's in the same row. Thus the sets of r t's in a row of the original array lie in an involution and $H_{r-1,r-1}=0$ is the condition that t,t' belong to the same set of the involution. By a similar argument the $\tau_{i,j}$'s in a row (i. e. with the same i) are sets of an involution which is determined by a form $\overline{H}_{r-1,r-1}=0$ symmetric in τ,τ'.

We consider then the resultant of $F_{r+1,r+1}$ and $\overline{H}_{r-1,r-1}$ in τ, a form of order r^2-1 in t and in τ' (which we henceforth call τ). This resultant R coördinates to $t=t_{i,j}$ the τ's of the row τ_{i+1} each taken $r-1$ times and the τ's of the row τ_i except $\tau_{i,j}$. Since $F_{r+1,r+1}$ coördinates to $t=t_{i,j}$ the τ's of the row τ_{i+1} and $\tau_{i,j}$ there follows that R and $F_{r+1,r+1}$ have a rational common factor $H_{r,r}$ which coördinates to $t=t_{i,j}$ the τ's of the row τ_{i+1} and $F_{r+1,r+1}$ has a residual factor $L_{1,1}$ which coördinates to $t=t_{i,j}$ the value $\tau=\tau_{i,j}$. Hence the sets of rs t's and rs τ's are projective under a projectivity (fixed

for the porism) which sends $t_{i,j}$ into $\tau_{i,j}$. If we take the parameter τ in such fashion that the projectivity is given by $t-\tau=0$ then $t_{i,j}=\tau_{i,j}$. Thus F has the factor $t-\tau$ and the residual factor $H_{r,r}$ is imprimitive, i. e., any t of a row determines the next row of τ's, now the same as the next row of t's. If then we write the involutions in which these sets lie as $(\gamma t)^r-l(\delta t)^r=0$, $(\gamma\tau)^r-\lambda(\delta\tau)^r=0$, the form $H_{r,r}$ becomes bilinear in l,λ. Moreover, since $H_{r,r}$ coördinates one row to the next this bilinear relation is cyclic and of period s and can be written as

$$\frac{l-f_1}{l-f_2}-\epsilon\frac{\lambda-f_1}{\lambda-f_2}=0,$$

where ϵ is a primitive sth root of unity. Hence $F_{r+1,r+1}$ has the canonical form

$$(11)\qquad (t-\tau)\left\{\frac{(\gamma t)^r-f_1(\delta t)^r}{(\gamma t)^r-f_2(\delta t)^r}-\epsilon\frac{(\gamma t)^r-f_1(\delta t)^r}{(\gamma t)^r-f_2(\delta t)^r}\right\}=0\,.$$

Conversely a form of this type is poristic with configurations of the type described. For if on a line ξ with coördinate λ,l on the same scale we indicate the s-ads of a cyclic collineation and then put the line ξ into $(1,r)$ correspondence with a line η with t,τ on the same scale, successive points of the original cyclic sets of s become successive rows of the original array. The addition of the factor $t-\tau$ then destroys the imprimitivity and opens a passage from the two successive sets to adjacent sets.

The involution $(\gamma t)^r-l(\delta t)^r=0$ has $2r-5$ absolute constants and these with f_1,f_2 give rise to $2r-3$ absolute constants in the poristic form. Thus we have proved that

(12) *For all values of $r\geqq 2$ and $s\geqq 2$ there exist poristic forms $F_{r+1,r+1}$ with configurations $\Delta^{r+1,r+1}_{rs,rs}$ of the cyclically imprimitive type described above. These forms contain $2r-3$ absolute constants.*

The case $r=2$ with a form $F_{3,3}$ and configurations $\Delta^{3,3}_{2s,2s}$ in ordinary space merits some consideration. The s pairs of points t lie on s bisecants of a cubic curve C^3 and the s bisecants are generators of one system of a quadric Q which contains C^3 since the s pairs are in an involution. A plane τ contains one pair and is therefore on a generator and is a tangent plane of Q. Thus the quadric Q as an envelope contains the planes of the cubic curve K^3. If λ,μ are the parameters of the two sets of generators such that the generators λ,μ meet in a point x of Q and lie in a plane ξ of Q, the parametric equations of Q as a point locus or locus of planes are

$$x_0=\lambda\mu\,,\quad x_1=\lambda\,,\quad x_2=\mu\,,\quad x_3=1\,;$$
$$\xi_0=1\,,\quad \xi_1=-\mu\,,\quad \xi_2=-\lambda\,,\quad \xi_3=\lambda\mu\,.$$

If the generators λ are bisecants of C^3 the equation of C^3 on Q is given by the $(1,2)$ form $(a\lambda)(b\mu)^2=0$. Since the generators λ cut out the pairs of points t in the configuration $\Delta^{3,3}_{2s,2s}$ the parameters λ of the s pairs are cyclo-projective and may be taken as $\lambda, \epsilon\lambda, \epsilon^2\lambda, \cdots, \epsilon^{s-1}\lambda$. The parameter on C^3 may be taken to be $t=\mu$. If then we take a point μ on C^3 the corresponding plane of K^3 is the plane on the same point and the preceding bisecant $\epsilon^{s-1}\lambda$, a plane which cuts Q in the generators $\epsilon^{s-1}\lambda, \mu$. If we solve for λ in $(a\lambda)(b\mu)^2=0$ and obtain $\lambda=(q\mu)^2/(r\mu)^2$ then we have for point $x(\lambda,\mu)$ on C^3 and plane $\xi(\epsilon^{s-1}\lambda,\mu)$ the coördinates

$$x_0=\mu(q\mu)^2\,,\quad x_1=(q\mu)^2\,,\qquad x_2=\mu(r\mu)^2\,,\quad x_3=(r\mu)^2\,;$$
$$\xi_0=\epsilon(r\mu)^2\,,\quad \xi_1=-\epsilon\mu(r\mu)^2,\quad \xi_2=-(q\mu)^2\,,\ \xi_3=\mu(q\mu)^2\,.$$

Then the incidence condition of plane $\xi(\mu')$ and point $x(\mu)$ is

$$(\mu-\mu')\left[\ \epsilon\frac{(q\mu)^2}{(r\mu)^2}-\frac{(q\mu')^2}{(r\mu')^2}\ \right]$$

which is of the same general form as (10). Hence

(13) *Given a cubic curve C^3 on a quadric Q; if C^3 is transformed into a cubic curve K^3 by a correlation which is the product of a collineation of period s which leaves every unisecant generator of C^3 unaltered and a polarity in Q, then C^3 and K^3 contain configurations $\Delta^{3,3}_{2s,2s}$. The bisecant generators of C^3 constitute an infinity of lines bisecant to C^3 which are also axes of K^3.*

Miss Cummings remarks that the double sextettes (β) furnish an infinity of such bisecant-axes which are not of the usual Hurwitz type. We see above that the case of an infinity of bisecant-axes for C^3, K^3 may arise in an infinity (for variable s) of ways, all projectively distinct. This, however, is not remarkable since C^3, K^3 are on the same quadric with one set of generators as bisecant-axes.

We think of the rs τ's arranged in an array like the t's, each τ being equal to the like-placed t. Then the form $H_{r,r}$ which appears as a factor in (11) and which we denote now by $H_{r,r}(\epsilon)$ correlates to any t all the τ's in the next row. A similar form $H_{r,r}(\epsilon^m)$ correlates to any t all the τ's in the row m times removed from the one in which t occurs. In this way for $m=1,\cdots,s-1$ any row of τ's is correlated to t except the one in which it occurs. The form $H_{r-1,r-1}$ correlates to t all the τ's in its own row except $\tau=t$, and the form $(t-\tau)$ correlates to t its own τ. It is evident then that the products

(14) $$(t-\tau)\,.\,H_{r,r}(\epsilon^{a_1})\,.\,\cdots\,.\,H_{r,r}(\epsilon^{a_j})\,,$$

(15) $$H_{r-1,r-1}\,.\,H_{r,r}(\epsilon^{a_1})\,.\,\cdots\,.\,H_{r,r}(\epsilon^{a_j})\,,$$

where $a_1, \cdots, a_j$ is a selection from the integers $1, \cdots, s-1$, are poristic and we see that

(16) *There exist poristic forms* $F_{jr+1,jr+1}$ *with configurations* $\Delta^{jr+1,jr+1}_{rs,rs}$; *and poristic forms* $F_{jr+r-1,jr+r-1}$ *with configurations* $\Delta^{jr+r-1,jr+r-1}_{rs,rs}$ *all of the cyclically imprimitive type where* $1 \leqq j \leqq s-1$.

In forming products like (14), (15) those with both factors $t-\tau, H_{r-1,r-1}$ are omitted since they are *imprimitive*, i. e., reducible to forms of lower degrees in t,τ by rational substitutions.

4. The poristic form for the double sextette (α)

Let $F_{3,3}$ be poristic with configurations $\Delta^{3,3}_{6,6}$ determined by the arrangement

(α) $$123 \,.\, 234 \,.\, 345 \,.\, 456 \,.\, 561 \,.\, 612$$

whose triads in the order given are denoted by $\tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_1$. The configuration may be regarded as a skew-hexagon in space with six ordered *vertices* $t_1, \cdots, t_6$, six ordered *sides* $t_1t_2, \cdots, t_6t_6, t_6t_1$ each joining two consecutive vertices, and six ordered planes $\tau_1 = 612, \cdots, \tau_5 = 456, \tau_6 = 561$ each on three consecutive vertices. The form $F_{3,3}=0$ determines the incidence of points and planes of the hexagon.

If t,t' determine in $F_{3,3}=0$ the same τ, i. e., if distinct vertices t,t' are on the same plane of the hexagon, they satisfy a symmetric form $G_{6,6}=0$ which coördinates to t_1 the values $t'=t_2,t_6$ each twice and t_3,t_5 each once. Thus $G_{6,6}$ has two rational factors $P^{(1)}_{2,2}$ and $P^{(2)}_{2,2}$ each symmetric in t,t'. The form $P^{(1)}_{2,2}$ is satisfied by adjacent vertices t,t'; the form $P^{(2)}_{2,2}$ is satisfied by vertices t,t' twice removed from each other. If we let the two values t' determined by t in $P^{(1)}_{2,2}$ be taken for t in $P^{(2)}_{2,2}$, the four values t' then obtained include the two already known from $P^{(1)}_{2,2}$ and two which are the vertices thrice removed from t and which are determined from a rationally known $P^{(3)}_{2,2}$. In the present case of a hexagon the vertices determined by $t=t_1$ in $P^{(3)}_{2,2}$ consist of t_4 taken twice, whence $P^{(3)}_{2,2}=(P_{1,1})^2$. Thus opposite vertices of the ∞^1 hexagons are in a quadratic involution, $P_{1,1}=0$.

By the same argument there exist forms symmetric in τ,τ': $R^{(1)}_{2,2}, R^{(2)}_{2,2}, R_{1,1}$, which for given τ determine respectively the adjacent planes, the planes twice removed and the opposite plane, of the hexagon with respect to the plane τ. Again the opposite planes of the ∞^1 hexagons are in a quadratic involution, $R_{1,1}=0$.

We observe that the three planes τ_6,τ_1,τ_2 on the vertex t_1 form an unsymmetrical triad in that τ_1 has a side in common with τ_6,τ_2 while τ_6,τ_2

have only the vertex t_1 in common. This indicates a factorization of the form $F_{3,3}$ into a product $Q_{1,1} \cdot Q_{2,2}$. The factors are rationally obtained as follows. The resultant in t of $P_{2,2}^{(1)}$ and $F_{3,3}$, of degree 6 in t' and in τ, coördinates to $t'=t_1$ the values τ_1 twice and $\tau_2,\tau_3,\tau_5,\tau_6$. Hence this resultant has a squared factor $Q_{1,1}$ and a residual quartic factor. Thus the factors $Q_{1,1},Q_{2,2}$ of $F_{3,3}$ are rationally known. The factor $Q_{1,1}$ correlates t_i to τ_i $(i=1,\cdots,6)$; the factor $Q_{2,2}$ correlates $t_1,\cdots,t_6$ in order to the following pairs of τ's: 62.13.24.35.46.51.

We make a change of variable τ so that $Q_{1,1}$ becomes $t-\tau$, i. e., every τ_i is equal to t_i. We also choose the involution $P_{1,1}$ to be $t'=-t$, whence $R_{1,1}$ is $\tau'=-\tau$. Then the values of the six t's and six τ's are

$$t_1,t_3,t_5=\tau_1,\tau_3,\tau_5=\lambda,\mu,\nu\ ;\quad t_4,t_6,t_2=\tau_4,\tau_6,\tau_2=-\lambda,-\mu,-\nu\ .$$

We observe that $Q_{2,2}$ correlates to t_1,t_3,t_5 the pairs formed from τ_4,τ_6,τ_2 and to t_4,t_6,t_2 the pairs formed from τ_1,τ_3,τ_5. Thus $Q_{2,2}$ admits two distinct configurations $\Delta_{3,3}^{2,2}$. If now we change the sign of τ then $Q_{2,2}$ is satisfied by any pair t,τ of λ,μ,ν or any pair t,τ of $-\lambda,-\mu,-\nu$, i. e., t,τ are roots of a cubic of the pencil $(a_0t^3+a_1t^2+a_2t+a_3)+r(a_0t^3-a_1t^2+a_2t-a_3)$ and $F_{3,3}$ has the form

$$(t+\tau)\left[\alpha t^2\tau^2+\beta(t^2+\tau^2)+\gamma t\tau+\delta\right]\ . \tag{17}$$

We observe that the configurations $\Delta_{6,6}^{3,3}$ of (17) are formed by tying together two configurations $\Delta_{3,3}^{2,2}$ of the factor $Q_{2,2}$ by means of the other factor $Q_{1,1}$.

5. Generalized Poncelet polygons

We consider a configuration $\Delta_{n,n}^{r,r}$ $(3\leqq r\leqq n-3)$* of n t's, $t_1,\cdots,t_n$ and n τ's, $\tau_1,\cdots,\tau_n$ such that each τ is coördinated to r of the t's by the following scheme:

$$123\cdots r\ .\ 234\cdots r,r+1\ .\ 34\cdots r+1,r+2\ .\ \cdots\ .\ n12\cdots r-1, \tag{18}$$

which admits the cyclic group $(12\cdots r\cdots n)$. This configuration will be called a "polygon in space S_r of r dimensions." We shall call $t_1,\cdots,t_n$ the *points* or vertices of the polygon; 12, 23, $\cdots$, $n1$ the *lines* or sides of the polygon; 123, 234, $\cdots$, $n12$ the *planes* of the polygon; and finally $12\cdots r,\cdots,n12\cdots r-1$ the S_{r-1}'s or *faces* of the polygon. Naturally such a polygon in S_r is self dual and is determined by the cyclic order of its n vertices t or of its n faces τ.

We first assume the existence of a poristic form $F_{r,r}$ which admits ∞^1 such configurations, find the necessary conditions which this imposes on

* For the cases $r=2$, $n-2$, $n-1$, cf. A. B. Coble, loc. cit., p. 17, 1°, 2°, 4°.

$F_{r,r}$, and finally show how these conditions can be satisfied and the poristic form actually constructed.

If distinct t,t' determine in $F_{r,r}$ a common τ, then t,t' satisfy a symmetric form G of order r^2-r in each. For given t_1, G determines the vertices t_2,t_n once removed from t_1 each $r-1$ times, the vertices t_3,t_{n-1} twice removed from t_1 each $r-2$ times, etc., whence

$$G=[P_{2,2}^{(1)}]^{r-1}[P_{2,2}^{(2)}]^{r-2}\cdots[P_{2,2}^{(r-1)}], \tag{19}$$

where $P_{2,2}^{(k)}=0$ is satisfied by vertices t,t' k times removed from each other. If $F_{r,r}$ is given, the form G and each of its factors is rationally known. Each of the forms $P^{(k)}$ is a rational covariant of $P^{(1)}$. For if we combine $P^{(1)}$ with $P^{(1)}$ (as $P^{(1)}$ was combined with $P^{(2)}$ in § 4 to produce $P^{(3)}$) we obtain $P^{(2)}$; and in general $P^{(1)}$ combined with $P^{(k-1)}$ produces $P^{(k)}$. Moreover if $F_{r,r}$ has the closure property, $P_{2,2}^{(1)}$ has the closure property for the n t's arranged cyclically; and $P_{2,2}^{(k)}$ has the closure property for the kth power of this cyclic arrangement.

(20) *The form $P_{2,2}^{(1)}$ is the form G associated with the planar Poncelet polygons of n sides determined by a $2,2$ form in t,σ.*

For if $P_{2,2}^{(1)}$ has the closure property for $t=t_1,t_2,\cdots,t_n$ we mark on a norm-conic C^2 with parameter t the closed sets of n points $t_1,\cdots,t_n$ and determine thereby ∞^1 planar polygons. For given $t=t_1$, adjacent vertices are given by $t'=t_n,t_2$ in $P_{2,2}^{(1)}=0$. The coördinates ξ of a line which cuts C^2 in t_1,t_2 are proportional to $1,t_1+t_2,t_1t_2$. Since $P_{2,2}^{(1)}$ is quadratic in these coördinates for $t,t'=t_1,t_2$, it is quadratic in ξ. Thus the sides ξ of the ∞^1 polygons envelop a conic K^2. If σ is any parameter on K^2 the incidence condition of C^2, K^2 is a $(2,2)$ form in t,σ of such sort that $P_{2,2}^{(1)}=0$ is the condition that t,t' are on the same side σ.

The same process of composition of $P_{2,2}^{(j)}$ with $P_{2,2}^{(1)}$ enables us to get any two vertices t' at like distance from t. Thus if n is odd, $n=2\nu+1$, the sequence of distinct forms P closes with

$$P_{2,2}^{(1)},\quad P_{2,2}^{(2)},\quad\cdots,\quad P_{2,2}^{(\nu)} \qquad (n=2\nu+1);$$

if however n is even, $n=2\nu$, the sequence of forms P is

$$P_{2,2}^{(1)},\quad P_{2,2}^{(2)},\quad\cdots,P_{2,2}^{(\nu)} \qquad (n=2\nu),$$

and $P_{2,2}^{(\nu)}$ is $(I_{1,1})^2$, i. e., the parameters of opposite vertices of the polygon are in a quadratic involution.

Similarly $F_{r,r}$ will determine a series of forms symmetric and of degrees 2,2 in τ,τ' comprising, for $n=2\nu+1$,

$$R_{2,2}^{(1)}, \quad R_{2,2}^{(2)}, \cdots, R_{2,2}^{(\nu)};$$

and, for $n=2\nu$,

$$R_{2,2}^{(1)}, \quad R_{2,2}^{(2)}, \cdots, R_{2,2}^{(\nu)},$$

where $R_{2,2}^{(\nu)}=(J_{1,1})^2$, i. e., the parameters of opposite faces of the polygon are in a quadratic involution.

Provisionally we refer to the r-ads of t's in the above scheme in the order given as $\tau_1, \cdots, \tau_n$ (later, for odd r at least, a much better choice can be made). Then the scheme of r-ads of τ's coördinated to $t_1, \cdots, t_n$ is

$$(21) \quad \tau_{n-r+2}\cdots\tau_n\tau_1\,.\,\tau_{n-r+3}\cdots\tau_n\tau_1\tau_2\,.\,\cdots\,.\,\tau_{n-r+1}\tau_{n-r+2}\cdots\tau_n.$$

We consider now the resultant in t of $P_{2,2}^{(1)}$ and $F_{r,r}$, a form of degree $2r$ in t' and τ. This resultant furnishes for $t'=t_1$ the values $\tau=\tau_{n-r+3}, \cdots, \tau_n$ each twice, and the pairs of values τ_{n-r+2}, τ_1 and τ_{n-r+1}, τ_2 each once. Hence there can be rationally factored from this resultant a form $F_{r-2,r-2}$ which coördinates to t_1 all the faces τ on t_1 which are coördinated to t_1 by $F_{r,r}$ except the extreme faces τ_{n-r+2}, τ_1. Thus $F_{r,r}$ has the rational factors $F_{r-2,r-2}$ and the residual factor $F_{2,2}^{(1)}$. If $F_{r,r}$ has the closure property each of these factors has the closure property and the $(r-2)$-ads and duads of τ's for these factors which are ascribed to $t_1, \cdots, t_n$ respectively are

$$(22) \quad \begin{array}{l} \tau_{n+r+3}\cdots\tau_n\,.\,\tau_{n-r+4}\cdots\tau_n\tau_1\,.\,\cdots\,.\,\tau_{n-r+2}\cdots\tau_{n-1}\,; \\ \tau_{n-r+2}\tau_1\,.\,\tau_{n-r+3}\tau_2\,.\,\cdots\,.\,\tau_{n-r+1}\tau_n\,. \end{array}$$

The factor $F_{r-2,r-2}$ is now poristic with polygons of n vertices in S_{r-2}. We may think of the original polygon in S_r as projected from a line upon an S_{r-2}. The S_{r-2}'s, and the S_{r-1}'s, or faces, of the original polygon are lost and the S_{r-3}'s of the original polygon become the *faces* of the projected polygon.

Again we form the resultant of $P_{2,2}^{(1)}$ and $F_{r-2,r-2}$ and obtain the square of a rational factor $F_{r-4,r-4}$ which coördinates to $t'=t_1$ the values $\tau_{n-r+4}\cdots\tau_{n-1}$. Thus $F_{r-2,r-2}$ factors into $F_{r-4,r-4}F_{2,2}^{(2)}$, two poristic forms whose scheme of τ's is

$$(23) \quad \begin{array}{l} \tau_{n-r+4}\cdots\tau_{n-1}\,.\,\tau_{n-r+5}\cdots\tau_n\,.\,\cdots\,.\,\tau_{n-r+3}\cdots\tau_{n+2}\,, \\ \tau_{n-r+3}\tau_n\,.\,\tau_{n-r+4}\tau_1\,.\,\cdots\,.\,\tau_{n-r+2}\tau_{n-1}\,. \end{array}$$

Continuing in this way to remove factors $F_{2,2}$ from $F_{r,r}$, which cuts off at each stage the extreme faces of the set on a given vertex, we finally wind up in one of two ways according as r is odd or even.

If $r=2\rho$ we have the following complete resolution of $F_{r,r}$:

$$F_{r,r}=F_{2,2}^{(1)}\cdot F_{2,2}^{(2)}\cdot\cdots\cdot F_{2,2}^{(\rho)}; \tag{24}$$

with the following scheme for the successive poristic forms:

$$\begin{array}{l}
\tau_{n-r+2}\tau_1\cdot\tau_{n-r+3}\tau_2\cdot\cdots\cdot\tau_{n-r}\tau_{n-1}\cdot\tau_{n-r+1}\tau_n\ ;\\
\tau_{n-r+3}\tau_n\cdot\tau_{n-r+4}\tau_1\cdot\cdots\cdot\tau_{n-r+1}\tau_{n-2}\cdot\tau_{n-r+2}\tau_{n-1}\ ;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\tau_{n-\rho}\tau_{n-\rho+3}\cdot\tau_{n-\rho+1}\tau_{n-\rho+4}\cdot\cdots\cdot\tau_{n-\rho-2}\tau_{n-\rho+1}\cdot\tau_{n-\rho-1}\tau_{n-\rho+2}\ ;\\
\tau_{n-\rho+1}\tau_{n-\rho+2}\cdot\tau_{n-\rho+2}\tau_{n-\rho+3}\cdot\cdots\cdot\tau_{n-\rho-1}\tau_{n-\rho}\cdot\tau_{n-\rho}\tau_{n-\rho+1}\cdot
\end{array} \tag{25}$$

If $r=2\rho+1$ we have the following complete resolution of $F_{r,r}$:

$$F_{r,r}=F_{2,2}^{(1)}\cdot F_{2,2}^{(2)}\cdot\cdots\cdot F_{2,2}^{(\rho)}\cdot F_{1,1}\ ; \tag{24'}$$

with the following scheme for the successive poristic forms:

$$\begin{array}{l}
\tau_{n-r+2}\tau_1\cdot\tau_{n-r+3}\tau_2\cdot\cdots\cdot\tau_{n-r}\tau_{n-1}\cdot\tau_{n-r+1}\tau_n\ ;\\
\tau_{n-r+3}\tau_n\cdot\tau_{n-r+4}\tau_1\cdot\cdots\cdot\tau_{n-r+1}\tau_{n-2}\cdot\tau_{n-r+2}\tau_{n-1}\ ;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\tau_{n-\rho}\tau_{n-\rho+2}\cdot\tau_{n-\rho+1}\tau_{n-\rho+3}\cdot\cdots\cdot\tau_{n-\rho-2}\tau_{n-\rho}\cdot\tau_{n-\rho-1}\tau_{n-\rho+1}\ ;\\
\tau_{n-\rho+1}\cdot\tau_{n-\rho+2}\cdot\cdots\cdot\tau_{n-\rho-1}\cdot\tau_{n-\rho}\cdot
\end{array} \tag{25'}$$

The $(2,2)$ forms $F_{2,2}$ in (24) and (24′) are each poristic if $F_{r,r}$ is poristic. Each one therefore defines a system of planar Poncelet polygons. But the cyclic arrangement in these various planar systems is not the cyclic arrangement $C=(123\cdots n)$ of the scheme (18). Indeed for the forms $F^{(1)},\cdots,F^{(\rho)}$ in (24) the cyclic arrangement is $C^{2\rho-1}, C^{2\rho-3},\cdots, C^3, C^1=C$ respectively; whereas for the forms $F^{(1)},\cdots,F^{(\rho)}$ in (24′) the cyclic arrangement is $C^{2\rho}, C^{2\rho-2},\cdots, C^4, C^2$.

Hence to construct a poristic form of the type (24) we begin with a form $F_{2,2}^{(\rho)}$ determined by *planar* Poncelet polygons on n sides and construct its "iterations" $F_{2,2}^{(\rho-1)},\cdots$. The rational process for this is as follows. We eliminate τ from $F_{2,2}^{(\rho)}(t)=0$ and $F_{2,2}^{(\rho)}(t')=0$ and, after separating the factor $(t-t')^2$ obtain $G_{2,2}^{(\rho)}(t,t')$, which coördinates adjacent vertices t,t'. The resultant in t of $G_{2,2}^{(\rho)}(t,t')$ and $F_{2,2}^{(\rho)}(t)$ is the product of $F_{2,2}^{(\rho)}$ and $F_{2,2}^{(\rho-1)}$, the original form and its iteration. The resultant in t of $G_{2,2}^{(\rho)}(t,t')$ and $F_{2,2}^{(\rho-1)}$ is the product of $F_{2,2}^{(\rho)}$ and $F_{2,2}^{(\rho-2)}$, etc. Obviously the form $F_{r,r}$ constructed in (24) with these factors is poristic with the configurations required.

For $r=2\rho+1$ there is, of the r vertices on a face, one central vertex isolated, i.e., every τ determines one t and vice versa. This projectivity between τ and t is given by the form $F_{1,1}$ in (24′), and we shall now choose the

order of the n τ's so that τ_i corresponds to t_i in this projectivity. Furthermore, we shall change the parameter τ in such a way that this projectivity is $t-\tau=0$, whence $\tau_i=t_i$. Then (25′) becomes

$$
\begin{array}{ccccccc}
\underline{t_1} & & \underline{t_2} & & \cdots & \underline{t_{n-1}} & \underline{t_n} \\
\tau_{n-\rho+1}\tau_{\rho+1} & & \tau_{n-\rho+2}\tau_{\rho+2} & & \cdots & \tau_{n-\rho-1}\tau_{\rho-1} & \tau_{n-\rho}\tau_{\rho} \\
\tau_{n-\rho+2}\tau_{\rho} & & \tau_{n-\rho+3}\tau_{\rho+1} & & \cdots & \tau_{n-\rho}\tau_{\rho-2} & \tau_{n-\rho+1}\tau_{\rho-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\tau_n\tau_2 & & \tau_1\tau_3 & & \cdots & \tau_{n-2}\tau_n & \tau_{n-1}\tau_1 \\
\tau_1 & & \tau_2 & & \cdots & \tau_{n-1} & \tau_n
\end{array}
\qquad (\tau_i=t_i). \tag{25''}
$$

Due to the coalescence of τ's with t's the form $F_{2,2}^{(\rho)}$ may now be regarded as a form which coördinates to t_1 the adjacent vertices τ_n, $\tau_2=t_n$, t_2 of a planar polygon of n sides. Hence if an $F_{2,2}(t,\tau)$ associated with planar Poncelet polygons of n sides is given we construct the symmetric form $F_{2,2}^{(\rho)}(t,t')$ which coördinates adjacent vertices and construct the iterations $F_{2,2}^{(\rho-1)}(t,t'), \cdots, F_{2,2}^{(1)}(t,t')$. If t' is replaced by τ in these forms and $F_{1,1}$ replaced by $t-\tau$, the form $F_{r,r}$ in (24′) is determined. Thus for $n=6$, $r=3$ we have the form (17) symmetric in t,τ. The further condition on this form —invariance under the simultaneous change of sign of t and τ—is due to the fact that n is even.

If now we interpret the form $F_{r,r}$ as the incidence condition of point t of a rational norm curve C^r in S_r with S_{r-1} of a rational norm curve K^r in S_r, we may state briefly the following theorem for the generalized Poncelet polygons:

(26) *For any value $n>2$ and $1<r<n$ there exist systems of ∞^1 Poncelet polygons P_n in S_r whose n vertices run over a rational norm curve C^r in S_r while the n faces envelop a rational norm curve K^r in S_r. The corresponding algebraic forms $F_{r,r}(t,\tau)$ with the closure property for configurations $\Delta_{n,n}^{r,r}$ of the type* (18) *are for $r>2$ factorable as indicated in* (24), (24′). *The factors of $F_{r,r}$ are rational covariants constructed as described above of the form $F_{2,2}(t,\tau)$ associated with the planar Poncelet polygons of n sides.*

6. Poristic forms $F_{3,3}$ with configurations $\Delta_{n,n}^{3,3}$ $(n<10)$

The poristic forms $F_{3,3}$ have a special interest in that the configurations which they admit can be visualized in ordinary space in association with the cubic curves C^3, K^3. If we look for possible poristic forms with configurations $\Delta_{n,n}^{3,3}$ our first task is the tabulation of arrangements of n triads τ of three out of n t's such that each t occurs in three triads. Here the concluding remark of §1 and more especially the transitivity requirement [(9)§1] are to be borne

in mind. A first question which then naturally arises is the following: Can a pair of t's which occurs in one triad τ occur again in one or more other triads? If t_1,t_2 occur in triads τ_1,τ_2, the line joining the points t_1,t_2 of C^3 is on the planes τ_1,τ_2 of K^3. This line is then both a bisecant of C^3 and an axis of K^3, say a "secant-axis" of C^3, K^3. Since a line cannot be on more than two planes of K^3 we see that

(27) *No pair of t's can occur in more than two triads τ.*

Such pairs as occur in two triads will be called *double pairs* of the arrangement.

A next question is as to the number of double pairs in which a given t may figure. Since the three planes of K^3 on a point t of C^3 meet in three axes of K^3 on t, clearly a given t cannot figure in more than three double pairs. If any double pairs occur, each t must figure in the same number and therefore at least in one. Then the curves C^3,K^3 would have ∞^1 secant axes if the form $F_{3,3}$ is poristic, and on a point t of C^3 there would be either 3, or 2, or 1 of these secant axes. As Miss Cummings' examples show, any one of these three cases may arise. We shall consider then the relative situation of two cubic curves C^3,K^3 which have ∞^1 secant-axes and the further specialization which arises when the form $F_{3,3}$ is poristic.

(a) *Through any point of C^3 there are three secant-axes of C^3,K^3.* Then the three planes of K^3 on a point t of C^3 meet in three secant-axes of C^3,K^3 on t which meet C^3 again at t_1,t_2,t_3 respectively. Then the three planes of K^3 on t_1 are tt_1t_2,tt_1t_3, and π. The plane π meets the other two planes in two bisecants of C^3 and these must be t_1t_2 and t_1t_3, else two bisecants of C^3 would meet in a point not on C^3, which is impossible if C^3 is a space cubic curve. Thus π is the plane t_1,t_2,t_3 and the four faces of the tetrahedron t,t_1,t_2,t_3 are planes of K^3. Hence the symmetric relation $(\beta t)^3(\beta' t')^3=0$ which expresses that tt' is a secant axis has one involutorial set t,t_1,t_2,t_3 and therefore has ∞^1 such sets which lie in an involution.

(28) *If C^3,K^3 have ∞^1 secant-axes, three on any point of C^3, then there exist ∞^1 tetrahedra with vertices on C^3 and faces on K^3 and the incidence condition $F_{3,3}=0$ is necessarily poristic with configurations $\Delta^{3,3}_{4,4}$.*

This case is referred to in (44) under the name of "Meyer's double quartettes."

(b) *Through any point of C^3 there are two secant-axes of C^3,K^3.* Then the three planes of K^3 on a point t of C^3 meet in two secant-axes tt_1,tt_2 and in a third line unisecant to C^3. The plane τ of the two secant-axes is a plane of K^3 projectively related to point t on the two secant-axes. If this projectivity

is taken to be $t=\tau$ and if $(\beta t)^2(\beta' t')^2=0$ is the symmetric relation on the parameters t,t' of points of C^3 on a secant axis, then the incidence condition of point t and plane τ is $(t-\tau)\cdot(\beta t)^2(\beta'\tau)^2=0$. If this $F_{3,3}$ is poristic then $(\beta t)^2(\beta'\tau)^2=0$ admits closure of the planar Poncelet polygon type. Indeed in space the common secant-axes form the sides of closed polygons.

(29) *If C^3,K^3 have ∞^1 secant-axes, two on any point of C^3, then there are two on any plane of K^3. The two curves C^3,K^3 are perspective, i.e., plane τ is on like-named point t and the incidence condition is $(t-\tau)\cdot(a\tau)^2(\alpha t)^2=0$ where $(a\tau)^2(\alpha t)^2$ is symmetric in t,τ. If further the $F_{3,3}$ form is poristic then C^3,K^3 admit Poncelet polygons of the type* [§5, $r=3$].

(c) *Through any point of C^3 there is one secant-axis of C^3,K^3.* The three planes of K^3 on t meet in three lines one of which is a secant-axis and the other two are unisecant to C^3. Thus a plane τ of K^3 contains one secant axis and meets C^3 in a further point t. Thus again C^3 and K^3 are perspective with a plane τ cutting out a secant-axis and $t=\tau$; and a point t lying on a secant-axis and a plane $\tau=t$. Points t,t' of a secant-axis satisfy a symmetric form $(\beta t)(\beta' t')=0$, i.e. lie in an $I_{1,1}$; whence the secant-axes are generators λ of a quadric Q containing the points of C^3 and the planes of K^3. The incidence condition of point t and plane τ is $(t-\tau)\cdot(a\tau)^2(\alpha t)^2=0$ where $(a\tau)^2(\alpha t)^2=0$ is *imprimitive* since each t of a pair on a secant-axis determines both τ's of a pair on that secant-axis and conversely.

(30) *If C^3,K^3 have ∞^1 secant-axes, one on any point of C^3, then also there is one on every plane of K^3. The curves C^3,K^3 are perspective and lie on the same quadric. The factor of the incidence condition residual to $t-\tau$ is imprimitive and coördinates the pairs t,t' and τ,τ' on the same secant-axis. If further the $F_{3,3}$ form is poristic then C^3,K^3 admit the cyclically imprimitive configurations of the type* [§3, (13)].

Since (28), (29), (30) exhaust the cases of arrangements with double pairs we have the following theorem:

(31) *All poristic forms $F_{3,3}$ with double pairs are included in the general aggregates of* §3 *and* §5.

Since for a given value of n the possible arrangements which are considered in §3 and §5 are easily tabulated we shall consider only configurations without double pairs. This materially reduces the number of possible configurations. For example, it is easily shown for $n=7$ that

(32) *The only poristic form $F_{3,3}$ with configurations $\Delta^{3,3}_{7,7}$ which have no double pairs is that of White.*

We examine now the possible configuration $\Delta_{8,8}^{3,3}$ without double pairs. Of the 28 possible pairs formed from 8 t's, 24 must appear in the 8 triads τ, and 4 pairs must be missing. Because of the transitivity each t must appear in one of the missing pairs. We take these missing pairs to be 12, 34, 56, 78. Since three of the triads must contain 1 and three others must contain 2, there remain two triads which contain neither 1 nor 2. These must each contain one figure out of each of the three other pairs and may be chosen to be 357.468. In one triad the pair 13 occurs and this triad cannot contain 5 or 7 since the pairs 35,37 already occur. The triad may contain 6 or 8 and since 68 occurs as a pair in the first two triads the choice of 6 or 8 is not material. Hence we choose a triad 136. The triad containing the pair 15 can contain also 4 or 8. Again the choice is not material and we select 158 and must then also select 147. The triad containing 23 must contain 8 since the pair 34 is not to occur and the pairs 35,36,37 already occur. Similarly we find, for the three triads containing 2, 238,245,267. Thus the only type of $\Delta_{8,8}^{3,3}$ without double pairs is

$$357 \,.\, 468 \,.\, 136 \,.\, 158 \,.\, 147 \,.\, 238 \,.\, 245 \,.\, 267 \,. \tag{33}$$

We shall call these triads in the order written

$$\tau = a, b, g, c, e, f, h, d \,,$$

so that we have isolated pairs of τ's (τ triads without a common t) ab, cd, ef, gh which correspond in order to the pairs of t's 12, 34, 56, 78 in the sense that the pair of τ's, ab, without a common t contain all the t's except the pair 12.

The group of the array (33) is of order 48 and abstractly identical with the product of a $g_{4!}$ (permutations of the four pairs) and an interchangeable g_2 defined by the element (12)(34)(56)(78). For the array is unaltered by (16472538), a cyclic element of period 4 on the four pairs; and by (357)(468), a cyclic element of period 3 on the four pairs. These generate $g_{4!}$. If every pair is unaltered and 1 is unaltered, then 2 is also unaltered. If now 3 in 136 interchanges with 4 in 147 then 6 must go into 7 and the pairs are altered. Hence (12)(34)(56)(78) is the only element other than the identity which leaves each pair unaltered and it is obviously invariant in the g_{48}.

Let us assume that there exists a poristic form $F_{3,3}$ with configurations $\Delta_{8,8}^{3,3}$ of the type (33). Then the form $G_{6,6}$ symmetric in t, t', which are in a common triad, coördinates to $t = t_1$ the values $t' = t_3, t_4, t_5, t_6, t_7, t_8$. The resultant in t of $G_{6,6}(t, t')$ and $F_{3,3}(t, \tau)$ coördinates to $t' = t_1$ the pair $\tau = a, b$ three times and the other three pairs each twice, whence this resultant has a rational factor $F_{2,2}(t', \tau)$ which coördinates to $t' = t_1$ or $t' = t_2$ the same pair

$\tau = a, b$ and vice versa. The form $G_{2,2}$ in t, t' formed for $F_{2,2}(t, \tau)$ becomes the square of a symmetric form $F_{1,1}(t, t')$, whence the pairs 12,34,56,78 are pairs of a quadratic involution and by a similar argument the pairs $ab, \cdots, gh$ are in involution. If we take the double points of these involutions to be $0, \infty$ then the four pairs of t's are determined by the equations

$$t^2 = l^2, m^2, n^2, r^2 ;$$

and the four pairs of τ's by the equations

$$\tau^2 = \lambda^2, \mu^2, \nu^2, \rho^2 .$$

Moreover, in $F_{2,2}$ the same pair of τ's is determined by either t of a pair, whence $F_{2,2}$ is bilinear in t^2 and τ^2, or the values l^2, m^2, n^2, r^2 and $\lambda^2, \mu^2, \nu^2, \rho^2$ have the same double ratio.

The resulting canonical form of $\Delta_{8,8}^{3,3}$ is

$$\begin{aligned} t_1, \cdots, t_8 &= l, -l, m, -m, n, -n, r, -r ; \\ \tau_a, \cdots, \tau_h &= \lambda, -\lambda, \mu, -\mu, \nu, -\nu, \rho, -\rho . \end{aligned} \tag{34}$$

If now we can find a form $F_{3,3}$ which admits this one configuration then $F_{3,3}$ is necessarily poristic. For it is easily verified from (34) and (33) that the form $F'_{3,3}$ obtained by changing the sign of τ in $F_{3,3}$ together with the form $F_{2,2}$ above combine to make an $F_{5,5}$ which admits the complementary configuration. If we examine the incidences for $F_{3,3}$ we see that $F_{3,3}$ must be unaltered when t, τ *both change sign*. Hence $F_{3,3}$ contains only 8 terms and 12 of the incidences imply the remaining 12. These 12 conditions on the 7 ratios of the coefficients of $F_{3,3}$ and the 8 constants $\lambda, \mu, \nu, \rho, l, m, n, r$ leave at least three degrees of freedom. But λ, l may each be taken as 1 by proper choice of the unit point for t, τ, so that there remains at least *one* degree of freedom for the form and its variable configuration. An attempt at this point, where the relative simplicity of the configuration (34) seemed favorable, to impose the algebraic conditions led to relations which were unmanageable and further geometric light on the required form $F_{3,3}$ seemed necessary.

Consider then the possible ways in which the $\Delta_{8,8}^{3,3}$ can degenerate by coincidences of the t's or τ's. Two planes τ can coincide only if some of the t's coincide. If two planes coincide they may be either (a) two planes of a pair such as $a = 357$, $b = 468$; or (b) two planes from different pairs as $g = 136$, $c = 158$. When a coincides with b the points 3,5,7 coincide with 4,6,8 and three cases arise according as there are three, one, or no coincidences in the pairs 34,56,78. Thus we have (a_1) $3 = 4, 5 = 6, 7 = 8$; (a_2) $3 = 4, 5 = 8, 6 = 7$; (a_3) $3 = 6, 5 = 8, 7 = 4$. The case (a_1) is impossible since the three planes

$a=357, c=157, d=257$ of K^3 are on a bisecant 57 of C^3. Also the case (a_3) cannot occur since 3,4 is one pair and 5,6 another pair of a quadratic involution, whence if 3 coincides with 6,4 must coincide with 5. Thus in case (a_3) the points 3,4,5,6,7,8 all would coincide. The case (a_2) leads to the coincidences $a=b, e=g, f=h$ among the planes, i.e., two planes of a pair and two other pairs coincide as with the points. The case (a_2) can happen just twice since the points of a pair can coincide only at $t=0$ and $t=\infty$. In case (b), $g=136\equiv C=158$, either (b_1) $3=5, 6=8$, or (b_2) $3=8, 5=6$. In the case (b_2) since $3=8$ then $4=7$ and with $5=6$ we have again case (a_2). In the case (b_1) we must have also $4=6, 5=7$, i.e. $3=5=7$ and $4=6=8$, i.e. three pairs coincide without a coincidence among the members of a pair. The configuration of planes behaves similarly since now $c=e=g$, $d=f=h$. Moreover, this case (b_1) can happen just twice. For in a pencil of octavics there are 14 coincidences among the roots, a k-fold root counting as $k-1$ coincidences. Thus the two instances of case (a_2) contribute 6 coincidences and each case (b_1) contributes 4 coincidences. With the help of these degenerate configurations we shall prove the following theorem:

(35) *There exists a poristic form $F_{3,3}$ containing one absolute constant which admits ∞^1 configurations $\Delta^{3,3}_{8,8}$ of the type* (33). *The eight points and eight planes on each configuration are distinct except in two cases each of types* (a_2) *and* (b_1) *above. This is the only poristic $F_{3,3}$ with double octettes other than that of the eight-sided Poncelet polygon in space and that of the cyclically imprimitive type which arises from a division of eight things into four sets of two.*

The equation $F_{3,3}=0$ is to be unaltered by the simultaneous change of sign of t and τ. It contains therefore terms $t^i\tau^j$ such that either $i+j$ is even or $i+j$ is odd. Either case may be reduced to the other by the substitution $\tau'=1/\tau$ which leaves the involution $\tau'=-\tau$ unaltered. We shall then take $F_{3,3}$ to be

$$\lambda_1\tau^3t^3+\lambda_2\tau^3t+\lambda_3\tau^2t^2+\lambda_4\tau^2+\lambda_5\tau t^3+\lambda_6\tau t+\lambda_7t^2+\lambda_8\ . \tag{36}$$

If now in a degenerate configuration of type (a_2) two points of a pair come together at $t=\infty$, then two planes of a pair come together at $\tau=0$. The two coincident pairs of points will be taken as $t=\pm1$; of planes as $\tau=\pm1$, whence all remaining constants are arbitrary. Then the scheme of parameter values of the eight t's and τ's for this degenerate configuration, when written as in (34), becomes

$$\begin{matrix}\infty,\infty,1,-1,1,-1,b,-b\ ,\\ \beta,-\beta,1,-1,-1,1,0,0\ .\end{matrix} \tag{34'}$$

The incidences implied by this scheme furnish just seven linear equations for the determination of $\lambda_1, \cdots, \lambda_8$ and the resulting values are

$$\begin{aligned}
\lambda_1 = -\lambda_5 &= (1-3\beta)\ , \\
\lambda_2 &= 2b^2(\beta-1)+(\beta+1)\ , \\
\lambda_3 &= (\beta+1)[b(\beta-2)+2(\beta-1)]\ , \\
\lambda_4 &= -(\beta+1)[2b(\beta-1)+(\beta-2)b]\ , \\
\lambda_6 &= 2b^2\beta(\beta-1)+(1+\beta)(1-2\beta)\ , \\
-\lambda_8 = \lambda_7 &= b\beta(\beta+1)\ .
\end{aligned}$$

The $F_{2,2}$ form bilinear in t^2, τ^2 must be satisfied by the pairs $t^2, \tau^2 = \infty$, β^2; $1, 1$; $b^2, 0$, whence it is

$$\tau^2 = \frac{\beta^2(t^2-b^2)}{t^2-(b^2\beta^2-\beta^2+1)}\ .$$

For the second degenerate configuration of type (a_2) the parameter scheme is

$$\begin{matrix} 0, 0, c, -c, c, -c, d, -d\ , \\ \delta, -\delta, \gamma, -\gamma, -\gamma, \gamma, \infty, \infty\ . \end{matrix} \tag{34''}$$

For $\tau = \infty$ and $t = 0$ the values of c^2 and γ^2 are found to be, in terms of b and β, the following:

$$c^2 = -\lambda_2/\lambda_1\ , \qquad \gamma^2 = \lambda_7/\lambda_4\ .$$

Also the $F_{2,2}$ form in t^2, τ^2 must be satisfied by the pairs $t^2, \tau^2 = 0, \delta^2$; d^2, ∞; c^2, γ^2. From the first two pairs we find

$$\delta^2 = b^2\beta^2/d^2\ , \qquad d^2 = b^2\beta^2-\beta^2+1\ .$$

The third pair c^2, γ^2, already known in terms of b, β, leads to the following condition on b, β and therefore leaves only one constant:

$$b = (1-2\beta)/\beta\ .$$

We now express all the quantities involved in terms of β as follows:

$$\begin{aligned}
\lambda_1 = -\lambda_5 &= \beta^2(1-3\beta)\ , & b &= (1-2\beta)/\beta\ , \\
\lambda_2 &= -d^2(1-3\beta)\ , & d^2 &= 3\beta^2-4\beta+2\ , \\
\lambda_3 &= \beta(\beta+1)(3\beta-2)\ , & \delta^2 &= (1-2\beta)^2/d^2\ , \\
\lambda_4 &= d^2(\beta+1)(1-2\beta)\ , & c^2 &= d^2/\beta^2\ , \\
\lambda_6 &= \beta(1-3\beta)(1-2\beta)(\beta-2)\ , & \gamma^2 &= \beta^2/d^2\ . \\
-\lambda_8 = \lambda_7 &= \beta^2(\beta+1)(1-2\beta)\ ;
\end{aligned} \tag{37}$$

The $F_{2,2}$ form in t^2,τ^2 now leads to

$$\tau^2=\frac{\beta^2(t^2-b^2)}{t^2-d^2}\,. \tag{38}$$

It is necessary to verify also the further incidences for the configuration (34''). When $t=c,\tau=\infty,\delta,\gamma$. The quadratic in τ with roots δ,γ obtained from (36), furnishes values

$$\delta+\gamma=c\frac{\lambda_1c^2-\lambda_6}{\lambda_3c^2+\lambda_4}=c\,\frac{\beta(3\beta-1)}{d^2}\,,$$

$$\delta\gamma=\frac{\lambda_7(c^2-1)}{\lambda_3c^2+\lambda_4}=\frac{\beta(2\beta-1)}{d^2}\,.$$

On squaring $\delta+\gamma$ and $\delta\gamma$ we find that these values are consistent with the relations (37). When $t=d,\tau=\delta,-\gamma,-\gamma$, from which we have in (36)

$$\delta-2\gamma=1/d\,,\qquad \gamma^2-2\gamma\delta=-\beta(3\beta-2)/d^2\,,\qquad \gamma^2\delta=\beta^2(1-2\beta)/d^3\,.$$

Again these values prove to be consistent with the relations (37). There remains the determination of the signs of c,γ,d,δ. Let

$$d=\sqrt{3\beta^2-4\beta+2}\,. \tag{39}$$

Then from (37) $\delta=\epsilon_1(1-2\beta)/d$, $c=\epsilon_2d/\beta$, $\gamma=\epsilon_3\beta/d$, where ϵ_1, ϵ_2, ϵ_3 are $+1$ or -1. From the values of $\delta+\gamma$, $\delta\gamma$, and $\delta-2\gamma$ just above, we find that

$$\delta=(1-2\beta)/d\,,\quad c=-d/\beta\,,\quad \gamma=-\beta/d\,. \tag{40}$$

Up to this point we have proved that

(41) *The form $F_{3,3}$ in* (36) *with coefficients* $\lambda_1,\cdots,\lambda_8$ *expressed in* (37) *as polynomials in the arbitrary constant β admits the two degenerate configurations* (34') *and* (34'') *with values c,γ,δ,d,b given in* (40), (39), *and* (37).

We still have to prove that this form $F_{3,3}$ is poristic. A study of the general configuration in (34) shows that the (5,5) form with configurations $\Delta^{5,5}_{8,8}$ which is *complementary* to $F_{3,3}$ with configurations $\Delta^{3,3}_{8,8}$ breaks up into a product of the form $F_{2,2}$ and a form $\bar F_{3,3}$ which arises from $F_{3,3}$ by a change of sign of τ alone. If now the product $F_{3,3}\cdot\bar F_{3,3}\cdot F_{2,2}$ can be expressed as a determinant $D_{8,8}$ as in (5) which is necessarily poristic then each factor such as $F_{3,3}$ is necessarily poristic. The form $F_{2,2}$ is obtained from (38),

$$F_{2,2}\equiv\tau^2t^2-d^2\tau^2-\beta^2t^2+d^2\delta^2\,.$$

Also

$$F_{3,3}\cdot\bar F_{3,3}\equiv\left[\lambda_3\tau^2t^2+\lambda_4\tau^2+\lambda_7(t^2-1)\right]^2-\tau^2t^2\left[\lambda_1t^2(\tau^2-1)+\lambda_2\tau^2+\lambda_6\right]^2\,.$$

For the determinant $D_{8,8}$ we know that the two octavics in t determined by the configurations (34′) and (34″) correspond respectively to the two octavics in τ similarly determined, so that a provisional form of $D_{8,8}$ is

$$D_{8,8}=\begin{vmatrix} (t^2-1)^2[\beta^2t^2-(1-2\beta)^2] & t^2(t^2-c^2)^2(t^2-d^2) \\ \mu^2\tau^2(\tau^2-\beta^2)(\tau^2-1)^2 & (\tau^2-\gamma^2)^2(\tau^2-\delta^2) \end{vmatrix}$$

where μ is a constant to be determined. We have to show that μ,ν can be determined so that

$$D_{8,8}=\nu F_{2,2}\cdot F_{3,3}\cdot \overline{F}_{3,3}\ .$$

A tedious direct calculation can be avoided here by the following argument. If we apply (38) regarded as a substitution on τ^2 to $D_{8,8}$ the second row becomes a numerical multiple of the first row provided that

$$\mu=\frac{\beta^4(1-3\beta)^2}{d^6(\beta+1)^2}\ .$$

With this value of μ, $D_{8,8}$ must contain the factor $F_{2,2}$. The residual factor of $D_{8,8}$ is a $(3,3)$ form $G_{3,3}(s,\sigma)$ in $s=t^2$ and $\sigma=\tau^2$ which by inspection is seen to be satisfied by the following pairs of values of s and σ:

$$\begin{array}{llll} s=\infty, & \sigma=1,1,0\ ; & s=0, & \sigma=\gamma^2,\gamma^2,\infty\ ; \\ s=1, & \sigma=1,0,\beta^2\ ; & s=c^2, & \sigma=\delta^2,\gamma^2,\infty\ ; \\ s=b^2, & \sigma=\beta^2,1,1\ ; & s=d^2, & \sigma=\delta^2,\gamma^2,\gamma^2\ . \end{array}$$

These values are sufficient to identify $G_{3,3}(s,\sigma)$ as a poristic form with configurations $\Delta^{3,3}_{4,4}$ (the two given here being degenerate) which arises geometrically from tetrahedra inscribed in one cubic curve and circumscribed to another. But the product $F_{3,3}\cdot\overline{F}_{3,3}$ is satisfied by the same values and therefore is a multiple of the residual factor $G_{3,3}(s,\sigma)$ of $D_{8,8}$. Hence $F_{3,3}$ is a poristic factor of $D_{8,8}$.

This completes the proof of the theorem (35) except for the verification of the existence of the two degenerate configurations of type (b_1). The two octavics in t found in $D_{8,8}$ have a jacobian of order 14 which must have the factors t, t^2-1, t^2-c^2, indicating the coincidences $0,\infty,\pm1,\pm c$ found in (34′) and (34″). The remaining octavic factor of this jacobian should be a perfect square indicating the four coincidences in triples found in the two configurations of type (b_1). If we take these two octavics in the form

$$(t^2-1)^2(t^2-b^2)\ , \qquad t^2(t^2-c^2)(t^2-d^2)$$

and pass to the parameter b by using $\beta=1/(b+2)$, in which case $c^2=2b^2+4b+3$, $d^2=c^2/(b+2)^2$, the octavic factor of the jacobian turns out to be

$$[t^4+2bt^2-bd^2(b+2)]^2\ .$$

We shall close with a demonstration that there are no "double nonettes" or configurations $\Delta_{9,9}^{3,3}$ for a form $F_{3,3}$ which are of particular interest. The cases obviously present according to the above discussion are first of all the Poncelet polygons of 9 sides in space. Secondly, for the cyclically imprimitive cases the $9\ t_i=\tau_i\ (i=1, \cdots, 9)$ must be arranged in three rows of three, and to a t in one row there must correspond the three τ's in the next row. This case is imprimitive and arises from a periodic collineation of period three by transformation of the third order.

There remain the cases in which the 9 triads of t's contain no repeated pair of t's. The triads then contain 27 pairs of t's and 9 pairs of t's are not found in the triads. Because of the transitivity of the triads each t must occur in two of the 9 pairs, and these may therefore be written in the cyclic order

$$12\ .\ 23\ .\ 34\ .\ 45\ .\ 56\ .\ 67\ .\ 78\ .\ 89\ .\ 91\ .$$

The group which permutes the triads must permute these isolated pairs and therefore is either the dihedral $g_{2.9}$ constructed for the cycle 123456789 or a transitive subgroup of it. The only transitive subgroup of $g_{2.9}$ is the cyclic g_9 and the triads must therefore admit this cyclic group.

The formation of the triads is now the problem of arranging the 27 diagonals of a polygon of 9 vertices $1, 2, \cdots, 9$ into 9 triangles three of which have a vertex at each vertex of the polygon. Moreover, the 9 triangles must permute cyclically under the cyclic g_9. Since 1 occurs with 9 and 2 in the omitted pairs it must occur with each of the other t's in some triad. There is therefore a triad $13x_1$, and on applying g_9 we have triads $13x_1$, $24x_2$, $35x_3$. The 9 triads cannot permute cyclically in three sets of three, since $35x_3$ permutes into $46x_4$ which cannot be $13x_1$. Hence they permute in a cyclic set of 9 which is

$$13x_1\ .\ 24x_2\ .\ 35x_3\ .\ 46x_4\ .\ 57x_5\ .\ 68x_6\ .\ 79x_7\ .\ 81x_8\ .\ 92x_9\ .$$

Since 1 and 3 already occur with 2, 4, 9 in the omitted pairs and with 5 and 8 in the triads the first triad must be 136 or 137. The two triangles hereby obtained have sides which in order skip 1, 2, 3 vertices or 1, 3, 2 vertices of the original polygon. Thus the two cases are not distinct, and we choose

$$136\ .\ 247\ .\ 358\ .\ 469\ .\ 571\ .\ 682\ .\ 793\ .\ 814\ .\ 925\ , \tag{42}$$

observing that the choice of 6 rather than 7 cuts the group of the triads from the dihedral $g_{2.9}$ to a cyclic g_9.

We denote the triads in order by $\tau=a, b, \cdots, g, h, i$, and assume that a poristic form $F_{3,3}$ exists which admits ∞^1 configurations of the type (42). The symmetric form $G_{6,6}$ in t, t', where t, t' is a pair in a τ triad, coördinates

to t all the vertices of the polygon except t itself and the two adjacent vertices. The resultant of G and $F_{3,3}$ coördinates to $t=t_1$ the values $\tau=c$ three times, $\tau=i$ once and $\tau=a,b,d,e,f,g,h$ each twice. Hence there is a form $F_{1,1}$ which coordinates t_1 and i; a form $F_{1,1}$ which coördinates t_1 and c; and a form which contains the factor $F_{3,3}$ and a residual factor $F_{4,4}$ which coördinates to t_1 the values $\tau=b,d,f,g$,

The significance of these factors is clear from the figure of the polygon and the 9 triangles. On fixing $\tau=a=136$ we have a triangle whose sides 13, 36, 61 skip respectively the vertices 2; 4, 5; 7, 8, 9. Thus $\tau=a$ uniquely determines $t=t_2$ as in $F_{1,1}$; also it uniquely determines $t=t_8$, the middle vertex of 7, 8, 9 as in $F'_{1,1}$; it also uniquely determines the adjacent pair 7, 9 by a form $F_{2,2}$; as well as the pair 4, 5 by a form $F'_{2,2}$. Hence $F_{4,4}$ factors into $F_{2,2}\cdot F'_{2,2}$. We write for convenience the table of values of these four forms, each poristic if $F_{3,3}$ is poristic:

	a	b	c	d	e	f	g	h	i
$F_{2,2}$:	7,9	8,1	9,2	1,3	2,4	3,5	4,6	5,7	6,8
$F'_{2,2}$:	4,5	5,6	6,7	7,8	8,9	9,1	1,2	2,3	3,4
$F_{1,1}$:	2	3	4	5	6	7	8	9	1
$F'_{1,1}$:	8	9	1	2	3	4	5	6	7

In $F_{2,2}, \tau=a$ determines $t=7, 9$; these determine in $F'_{2,2}$, $\tau=c, d, e, f$; and these determine in $F_{2,2}$, $t=2, 3$ each twice and $t=1, 4, 5, 9$ each once. Hence there is a $2,2$ form which coördinates to $\tau=a$ the values $t=2,3$ and this factors into $F_{1,1}\cdot F''_{1,1}$, where $F''_{1,1}$ coördinates $\tau=a$ to $t=3$. Then $t=2$ in $F_{1,1}$ determines $\tau=a$ and this in $F''_{1,1}$ determines $t'=3$, whence there is a cyclic collineation which permutes $t=t_1, \cdots, t=t_9$ cyclically. If we take the fixed points of this collineation to be $0, \infty$, the parameters of the 9 t's are

$$k, k\epsilon, k\epsilon^2, \cdots, k\epsilon^8 \qquad (\epsilon \text{ a primitive root of unity}).$$

If we take the parameter system for τ in such wise that the form $F_{1,1}$ is $\tau-t$ then the parameters of the 9 τ's in order are

$$k\epsilon, k\epsilon^2, k\epsilon^3, \cdots, k\epsilon^8, k\ ;$$

and the form $F_{3,3}$ is merely

$$F_{3,3}=(t-\epsilon^3\tau)\cdot(t-\epsilon\tau)\cdot(t-\epsilon^4\tau)\ . \tag{43}$$

Such a product of positive powers of a cyclic collineation is obviously poristic.

This completes the survey of possible configurations $\Delta^{3,3}_{n,n}$ up to $n=10$ and leads to the following noteworthy theorem:

(44) *Of all possible poristic forms* $F_{3,3}$ *with configurations* $\Delta_{n,n}^{3,3}(n<10)$ *there are but three which do not degenerate into products of poristic forms of lower orders. The attached configurations are the double quartettes of Meyer, the double septettes of White, and the double octettes described in* (35).

For $n=10$ the group problem of tabulating possible poristic arrangements is itself not a simple one.

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ON A CLASS OF POLYNOMIALS IN THE THEORY OF BESSEL'S FUNCTIONS*

BY

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1. Those values of z for which Bessel's function

$$J_n(z)=\left(\frac{z}{2}\right)^n\left\{\frac{1}{\Gamma(n+1)\,\Gamma(1)}-\frac{\left(\frac{z}{2}\right)^2}{\Gamma(n+2)\,\Gamma(2)}+\cdots\right\}$$

vanishes are known to be infinite in number for a general value of n. Their importance in mathematical physics has led to their calculation for positive integral values of n. They may also be regarded as branches of an infinitely many branched function of a complex variable. From this point of view they have been studied but little, and even for the case of real values of n few results are known. For example, it has been proved that if $n>-1$ the roots are all real, and that each root is an increasing function of n if $n>0$.

From the recursion relations between the functions $J_n(z)$ for consecutive values of n a set of polynomials is derived which has been considered incidentally in the theory. Hurwitz† made a systematic study of these polynomials regarded as functions of z. These polynomials are also polynomials in n but have not been examined as such. In what follows some properties showing their character as functions of n are deduced, and it is found that these properties are susceptible of some applications. In particular, it is shown that the roots of $J_n(z)=0$ are increasing functions of n when n lies between -1 and 0, a result that cannot be deduced from Poisson's integral formula employed by Schläfli in deriving the similar conclusion for $n>0$.

2. In this paper the notation and results of Hurwitz are used and the following extract gives what is essential for its comprehension.

If

$$f_n(z)=\frac{1}{\Gamma(n+1)\,\Gamma(1)}+\frac{z}{\Gamma(n+2)\Gamma(2)}+\cdots$$

then

$$J_n(z)=\left(\frac{z}{2}\right)^n f_n\left(-\frac{z^2}{4}\right).$$

* Presented to the Society, February 24, 1906; received by the editors in January, 1925.

† A. Hurwitz, Mathematische Annalen, vol. 33 (1889), p. 246.

Between the functions f_n subsists the relation $f_n = g_p f_{n+p} + z h_p f_{n+p+1}$ where g_p and h_p are polynomials in z and n. It is found that $h_{p+1} = g_p$ and $g_{p+1} = (n+p+1)g_p + z g_{p-1}$. It is also found that

$$\frac{g_p}{\Gamma(n+p+1)} = f_n\left[1 + \frac{z}{-n-p} + \cdots\right] + \frac{(-1)^{p+1}\pi}{\sin(n+1)\pi} \cdot f_{-n} \cdot \frac{z^{p+1}}{\Gamma(n+p+1)\,\Gamma(n+p+2)}\left(1 + \frac{z}{n+p+2} + \cdots\right),$$

$n \neq$ integer.

In any region of the plane of z, $f_n(z)$ is the uniform limit of $g_p(z)/\Gamma(n+p+1)$. In accordance with a more general theorem proved by Hurwitz the roots of $f_n(z) = 0$ are the values given by the derived set of the set of roots of $g_p(z) = 0$.

If Δ_p denotes the Jacobian

$$g_p \frac{dg_{p+1}}{dz} - g_{p+1}\frac{dg_p}{dz}$$

there is the recurrence relation

$$\Delta_{p+2} = (n+p+2)g_{p+1}^2 + z^2\Delta_p$$

and Δ_p is found to be equal to

$$\left(\frac{\pi}{\sin(n+1)\pi}\right)^2\left(\frac{1}{\Gamma(-n-p-1)}\right)^2\left[\frac{1}{n+p+1} + \cdots\right].$$

If $f_n(z) = 0$, $\Delta_p = (-z)^{p-1} + \cdots$ where the terms not written tend to zero as p tends to infinity. From these equations, making use of the Heine-Borel theorem, it follows, if z is restricted to a given segment of the real axis, that m may be chosen so great that when $p > m$, $\Delta_{2p-1} > 0$. If $n > -1$ the roots of the equation $g_p = 0$ are all real and $\Delta_p > 0$ on any segment of the negative part of the real axis.

3. The functions g_i are polynomials in n as well as z, and the consideration of them as such leads to a number of new results. Let

$$D_p = g_p\frac{dg_{p+1}}{dn} - g_{p+1}\frac{dg_p}{dn};$$

then

$$D_p = g_p^2 - zD_{p-1}.$$

Differentiating $g_{p+1} = (n+p+1)g_p + z g_{p-1}$ with respect to n gives

$$\frac{dg_{p+1}}{dn} = g_p + (n+p+1)\frac{dg_p}{dn} + z\frac{dg_{p-1}}{dn}$$

and combining with

$$\frac{dg_p}{dn} = \frac{dg_p}{dn}$$

it follows that

$$g_p \frac{dg_{p+1}}{dn} - g_{p+1} \frac{dg_p}{dn} = g_p^2 - z\left(g_{p-1} \frac{dg_p}{dn} - g_p \frac{dg_{p-1}}{dn}\right)$$

or $D_p = g_p^2 - zD_{p-1}$, which is the required relation.

Since $D_0 = 1$, $D_1 = (n+1)^2 - z$ it follows that if a negative value is assigned to z, $D_p = g_p^2 + |z| D_{p-1} > 0$ so that for every value of n, $D_p > 0$ and g_{p+1}/g_p is an increasing function of n.

From this property of g_{p+1}/g_p it follows, if z_1 is a negative root of $g_{p+1} = 0$, and p is sufficiently large, that $g_p(z_1) > 0$ or < 0 according as g_{p+1} changes from negative to positive or the reverse when z increases through z_1. For by the results of Hurwitz if $z_1 < 0$ it may be shown that $\Delta_p > 0$ for the value z_1 of z and, p being sufficiently large, g_{p+1}/g_p is an increasing function of z. Writing $g_{p+1} = g_{p+1}(z, n)$ and taking n' a value slightly greater than n, it follows that $g_{p+1}(z_1, n') > 0$ in the first case and < 0 in the second. This is evident because in both cases

$$\frac{g_{p+1}(z_1, n')}{g_p(z_1, n')} > \frac{g_{p+1}(z_1, n)}{g_p(z_1, n)} = 0,$$

since $z_1 < 0$ and $g_{p+1}(z_1)/g_p(z_1)$ is an increasing function of n. If z_1' denotes the root of $g_{p+1}(z, n') = 0$ which differs slightly from z_1 it follows in both cases that $z_1' < z_1$.

According to a general theorem of Hurwitz the roots of $f_n = 0$ are the limits of the roots of the polynomials $g_{p+1}(z, n) = 0$. It follows from the preceding discussion, if r_k denotes the kth root of $f_n = 0$ and r_k' the kth root of $f_{n'} = 0$, that $r_k' \leqq r_k$. The equality may be excluded since the functions f_n and $f_{n'}$ cannot have a common root for a range of values of n.

Taking account of the relation between f_n and J_n, the inequality $r_k' < r_k$ gives the following theorem: *The absolute values of the real roots of* $J_n = 0$ *are increasing functions of n, whatever real value n may have. In particular if* $n > -1$ *the roots of* $J_n = 0$ *are all real and increase in absolute value with n.* It seems that this property of the roots cannot be deduced from the more familiar methods of treating such a question; these methods are, in fact, restricted to the range $n > 0$.

4. Another property of the functions g_p is found as follows. Differentiate the equation

$$f_n = g_p(z, n) f_{n+p} + z g_{p-1}(z, n) f_{n+p+1}$$

with respect to z and replace f_{n+p} by $(n+p+1)f_{n+p+1}+z\,f_{n+p+2}$. The result is

$$f_{n+1}=g_p(z,n)f_{n+p+1}+zg_{p-1}(z,n)f_{n+p+2}+g_{p-1}(z,n)f_{n+p+1}$$
$$+z\frac{dg_{p-1}(z,n)}{dz}f_{n+p+1}+\frac{dg_p(z,n)}{dz}[(n+p+1)f_{n+p+1}+zf_{n+p+2}]\,.$$

Since $f_{n+1}=g_p(z,n+1)f_{n+p+1}+z\,g_{p-1}(z,n+1)f_{n+p+2}$ it follows that

$$g_p(z,n+1)=g_p(z,n)+(n+p+1)\frac{dg_p(z,n)}{dz}+\frac{d}{dz}[zg_{p-1}(z,n)]$$

and

$$g_{p-1}(z,n+1)=g_{p-1}(z,n)+\frac{dg_p(z,n)}{dz}\,.$$

The first of these equations reduces to the second which may be retained in the form

$$g_p(z,n+1)-g_p(z,n)=\frac{dg_{p+1}(z,n)}{dz}\,.$$

An application of this equation may be made to the calculation of the roots of $f_n=0$ when $n>-1$. Let z_1, z_2, z_3 be the kth roots respectively of $g_{p+1}(z,n), g_p(z,n), g_p(z,n+1)$; then $z_1>z_2>z_3$ and the functions all change in the same sense from negative to positive or the reverse when z passes through the root corresponding to the function. If z' is the kth root of

$$\frac{dg_{p+1}(z,n)}{dz}=0$$

it follows from the equation

$$g_p(z,n+1)-g_p(z,n)=\frac{dg_{p+1}(z,n)}{dz}$$

that $g_p(z',n+1)=g_p(z',n)$ and it can be seen that

$$\frac{dg_{p+1}(z_2,n)}{dz}>0 \text{ or } <0$$

according as $g_{p+1}(z,n)$ changes from negative to positive or the reverse when z passes through z_1. This allows a series of approximations to be made to any root of $f_n=0$, say the first, as follows: Let z_2 be the root of $g_2=0$; form the quantities

$$z_3=z_2-\frac{g_3(z_2)}{g_3'(z_2)}\,,\qquad z_4=z_3-\frac{g_4(z_3)}{g_4'(z_3)}\,,\text{ etc.}$$

For example, if $n=0$, $z_4=-29/20$ which gives for the least positive root of $J_0=0$ the value 2.40. These approximations to the first root are rational functions of n, the value of z_4 being

$$-\frac{1}{2}(n+1)(n+3)+\frac{(n+1)^3}{4(2n+5)}.$$

The consideration of the convergence of this process of approximation is waived.

5. A class of arithmetical equations may be deduced from the equation

$$g_p(z,n+1)-g_p(z,n)=\frac{dg_{p+1}(z,n)}{dz}.$$

Omitting z from the notation for g_p and distinguishing p even from p odd,

$$\begin{aligned}g_{2p}(n)=z^p&+{}_{p+1}C_{p-1}(n+p+1)(n+p)z^{p-1}+\cdots\\&+{}_{p+r}C_{p-r}(n+p+r)\cdots(n+p-r+1)z^{p-r}+\cdots\\&+(n+2p)\cdots(n+1).\end{aligned}$$

Let $g_{2p}(n)$ be expanded in powers of n so that

$$g_{2p}(n)=\sum_{l=0\cdots 2p}B_l n^l;$$

then

$$\begin{aligned}B_{2p-2k}=&\sum_{1\cdots 2p}i_1\cdots i_{2k}+(2p-1)\sum_{2\cdots 2p-1}i_1\cdots i_{2k-2}z+\cdots\\&+{}_{p+r}C_{p-r}\sum_{p-r+1\cdots p+r}i_1\cdots i_{2k-2(p-r)}z^{p-r}+\cdots+{}_{2p-k}C_k z^k,\\B_{2p-(2k-1)}=&\sum_{1\cdots 2p}i_1\cdots i_{2k-1}+(2p-1)\sum_{2\cdots 2p-1}i_1\cdots i_{2k-3}z+\cdots\\&+{}_{p+r}C_{p-r}\sum_{p-r+1\cdots p+r}i_1\cdots i_{2k-2(p-r)-1}z^{p-r}+\cdots+{}_{2p-k+1}C_{k-1}\sum_{k\cdots 2p-k+1}i_1 z^{k-1}\end{aligned}$$

where

$$\sum_{1\cdots 2p}i\cdots i_{2k}$$

denotes the sum of the products of the numbers $1\cdots 2p$ taken $2k$ at a time. Similarly

$$\begin{aligned}g_{2p+1}(n)=&{}_{p+1}C_p(n+p+1)z^p+\cdots+{}_{p+r+1}C_{p-r}(n+p+r+1)\\&\cdots(n+p-r+1)z^{p-r}+\cdots+(n+2p+1)\cdots(n+1).\end{aligned}$$

Expanding as before it is found that

$$B_{2p-2k}=\sum_{1\cdots 2p+1} i_1\cdot\cdot\cdot i_{2k+1}+{}_{2p}C_1\sum_{2\cdots 2p} i_1\cdot\cdot\cdot i_{2k-1}z+\cdot\cdot\cdot$$
$$+{}_{p+r+1}C_{p-r}\sum_{p-r+1\cdots p+r+1} i_1\cdot\cdot\cdot i_{2k-2(p-r)+1}z^{p-r}+\cdot\cdot\cdot$$
$$+{}_{2p+1-k}C_k\sum_{k+1\cdots 2p+1-k} i_1z^k,$$

$$B_{2p-(2k-1)}=\sum_{1\cdots 2p-1} i_1\cdot\cdot\cdot i_{2k}+{}_{2p}C_1\sum_{2\cdots 2p} i_1\cdot\cdot\cdot i_{2k-2}z+\cdot\cdot\cdot$$
$$+{}_{p+r+1}C_{p-r}\sum_{p-r+1\cdots p+r+1} i_1\cdot\cdot\cdot i_{2k-2(p-r)}z^{p-r}+\cdot\cdot\cdot+{}_{2p+1-k}C_kz^k\,.$$

It may be supposed that $g_{2p}(n)$ is expanded in terms of Bernoulli's polynomials

$$g_{2p}(n)=a_0+a_1\phi_1+\cdot\cdot\cdot+a_{2p}\phi_{2p}\,,$$

where $\phi_1=n$, $\phi_2=\frac{1}{2}\,n(n+1)$, etc., and

$$a_0=z^p+\left[\frac{(p+1)p}{1\cdot 2}\right]^2z^{p-1}+\cdot\cdot\cdot+\left[\frac{(p+2)\cdot\cdot\cdot(p-r+1)}{1\cdot 2\cdot\cdot\cdot 2r}\right]^2z^{p-r}$$
$$+\cdot\cdot\cdot+[2p(2p-1)\cdot\cdot\cdot 1].$$

The polynomials ϕ_l satisfy the relations $\phi_l(n+1)-\phi_l(n)=n^{l-1}$ so that $g_{2p}(n+1)-g_{2p}(n)=a_1+a_2n+\cdot\cdot\cdot+a_{2p}n^{2p-1}$ and since

$$g_{2p}(n+1)-g_{2p}(n)=\frac{dg_{2p+1}(n)}{dz}$$

it follows that $a_l=dB_l/dz$; hence the coefficients of a_l may be found from the above values of B_l, those being selected which belong to the function g_{2p+1}.

Denoting by a_l^q the coefficient of z^q in a_l, and equating coefficients of the qth power of z in the equation

$$g_{2p}(n)=a_0+a_1\phi_1+\cdot\cdot\cdot+a_{2p}\phi_{2p}\,,$$

it follows that

$${}_{2p-q}C_q(n+2p-q)\cdot\cdot\cdot(n+q+1)=a_0^q+a_1^q\phi_1+\cdot\cdot\cdot+a_{2p}^q\phi_{2p}$$

where

$$a_0^q=\frac{[(2p-q)\cdot\cdot\cdot(q+1)]^2}{1\cdot 2\cdot\cdot\cdot(2p-2q)}\,,$$

. . . . . .

$$a^q_{2l} = (q+1)_{2p-q-1}C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdot \cdot \cdot i_{2p-2q-2l-2} ,$$

$$a^q_{2l-1} = (q+1)_{2p-q-1}C_{q+1} \sum_{q+2 \cdots 2p-q-1} i_1 \cdot \cdot \cdot i_{2p-2q-2l-1} .$$

The coefficients B_{2l}, B_{2l-1} involve powers of z up to $p-l$ so that a^q_{2l}, $a^q_{2l-1} = 0$ if $l > p-q-1$. When the preceding equation is reduced by excluding a factor of both sides it becomes

$$(2p-q)(n+2p-q) \cdot \cdot \cdot (n+q+1) = (2p-q)(2p-q)(2p-q-1) \cdot \cdot \cdot (q+1) + (2p-2q)(2p-2q-1)[b_1\phi_1 + \cdot \cdot \cdot + b_{2p}\phi_{2p}] ,$$

where b_l denotes the summation $\sum$ in a^q_l. By assigning special values to n, or by comparing coefficients of powers of n, or by taking the equation as a congruence, special results may be derived. For example, if $n=1$

$$(2p-q)(2p-q+1) \cdot \cdot \cdot (q+2) = (2p-q)(2p-q)(2p-q-1) \cdot \cdot \cdot (q+1) + (2p-2q)(2p-2q-1)[b_1 + \cdot \cdot \cdot + b_{2p}]$$

since if $n=1$, $\phi_l = 1$. If $2p-q$ is prime to $(2p-2q)$ $(2p-2q-1)$ and n is an integer, then

$$b_1\phi_1 + \cdot \cdot \cdot + b_{2p}\phi_{2p} \equiv 0 \pmod{(2p-2q)} , \text{ etc .}$$

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ON THE ZEROS OF THE FUNCTION $\beta(z)$ ASSOCIATED WITH THE GAMMA FUNCTION*

BY

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1. Nielsen† has raised the question whether the function $\beta(z)$ defined by

$$\beta(z)=\frac{1}{2}\left[\psi\left(\frac{1+z}{2}\right)-\psi\left(\frac{z}{2}\right)\right], \qquad \left(\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}\right),$$

has any zeros, and has shown that there are no real zeros, and that the complex zeros, if any, must have their real part less than $-\frac{1}{2}$.

This question will be answered completely in the following, by showing that

The zeros of $\beta(z)$ are all complex, and their real part less than $-\frac{1}{2}$. For $n=1, 2, 3, \cdots$, each of the infinite strips

$$-2n-\frac{1}{2}<\textit{real part of } z<-2n+\frac{3}{2}$$

contains exactly two zeros, and for n sufficiently large, their asymptotic expression is

$$-2n+\frac{1}{2}+\frac{2\log(8n+2)\pi}{(4n+1)\pi^2}\pm i\frac{\log(8n+2)\pi}{\pi}+\frac{\xi_n\pm i\eta_n}{2\pi n},$$

$$\xi_n^2+\eta_n^2<1.$$

2. From the definition of $\beta(z)$ it follows at once that

$$(1)\qquad \beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{z+n}=-\log 2+\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z+n}-\frac{1}{n}\right),$$

both series converging uniformly (and the second also absolutely) in any finite region in the z-plane to which the poles $z=0, -1, -2, \cdots$ are exterior; and from (1) (or directly from the definition) we obtain the relation

$$(2)\qquad \beta(z)+\beta(1-z)=\frac{\pi}{\sin \pi z}.$$

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* Presented to the Society, September 5, 1916; received by the editors in April, 1925.

† N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Leipzig, Teubner, 1906. See p. 101.

Let us begin by proving Nielsen's results. For z real and positive, the terms in (1) have alternating signs and decrease in absolute value, so that $\beta(z)>0$. For $z=-m-\zeta$, where m is a positive integer or zero, and $0<\zeta<1$, we may write (1) in the form

$$(-1)^{m+1}\beta(-m-\zeta)=-\left[\frac{1}{1+\zeta}-\frac{1}{2+\zeta}+\cdots+\frac{(-1)^m}{m+\zeta}\right]+\frac{1}{\zeta}$$
$$+\frac{1}{1-\zeta}-\left[\frac{1}{2-\zeta}-\frac{1}{3-\zeta}+\cdots\right];$$

the terms in brackets having alternating signs and decreasing in absolute value, we have

$$(-1)^{m+1}\beta(-m-\zeta)>\frac{1}{\zeta}+\frac{1}{1-\zeta}-\frac{1}{1+\zeta}-\frac{1}{2-\zeta}$$
$$=\frac{1}{\zeta(1+\zeta)}+\frac{1}{(1-\zeta)(2-\zeta)}>0\,.$$

Hence there are no real zeros. On the other hand, let $z=x+yi$, $y\neq 0$, be a complex zero; taking the imaginary part of (1) and dividing by y, we find

$$\frac{1}{x^2+y^2}-\frac{1}{(x+1)^2+y^2}+\left[\frac{1}{(x+2)^2+y^2}-\frac{1}{(x+3)^2+y^2}+\cdots\right]=0\,,$$

and assuming $x>-2$, the terms in the bracket have alternating signs and decrease in absolute value, whence

$$\frac{1}{x^2+y^2}-\frac{1}{(x+1)^2+y^2}<0 \quad \text{or } x<-\frac{1}{2}\,.$$

3. It is convenient to determine first the zeros of $\beta(1-z)$ and then replace z by $1-z$. From the preceding results it follows that the zeros of $\beta(1-z)$ will be complex and have their real part greater than $\frac{3}{2}$. To obtain an asymptotic expression for $\beta(1-z)$, we observe that, for $\Re z>0$ ($\Re z=x=$ real part of z), the expression (1) may be replaced by

$$\beta(z)=\int_0^\infty \frac{e^{-zu}}{1+e^{-u}}\,du\,,$$

whence, integrating twice by parts,

$$\beta(z)=\frac{1}{2z}+\frac{1}{4z^2}-\frac{1}{z^2}\int_0^\infty e^{-zu}\,\frac{e^{-u}-e^{-2u}}{(1+e^{-u})^3}\,du\,. \tag{3}$$

For $\Re z \geqq 1$, we have

$$\left| \int_0^\infty e^{-zu} \frac{e^{-u}-e^{-2u}}{(1+e^{-u})^3}\, du \right| < \int_0^\infty e^{-u}(e^{-u}-e^{-2u})du = \frac{1}{6},$$

and (2) and (3) now give the asymptotic expression

$$\beta(1-z) = \frac{\pi}{\sin \pi z} - \frac{1}{2z} - \frac{h}{z^2}, \qquad |h| < \frac{5}{12} \text{ for } \Re z \geqq 1. \tag{4}$$

4. Let us consider first the zeros of the expression

$$\frac{\pi}{\sin \pi z} - \frac{1}{2z}, \tag{5}$$

which constitutes the principal part of $\beta(1-z)$. Since $\sin \pi(x+yi) = \sin \pi x \operatorname{ch} \pi y + i \cos \pi x \operatorname{sh} \pi y$, a zero $z=x+yi$ of (5) implies the two equations

$$\sin \pi x \operatorname{ch} \pi y = 2\pi x, \tag{6}$$

$$\cos \pi x \operatorname{sh} \pi y = 2\pi y, \tag{7}$$

where, however, the solution $x=y=0$ must be discarded, since it corresponds to the pole $z=0$ of (5). It is evident that when $x+yi$ is a zero of (5), the same is true of $x-yi$, $-x+yi$ and $-x-yi$; we may therefore restrict the discussion of (6) and (7) to the case $x \geqq 0$, $y \geqq 0$. First assume $x=0$; since $y \geqq 0$ in this case, and $(\operatorname{sh} \pi y)/\pi y$ increases steadily from 1 to ∞ as y increases from 0 to ∞, (7) gives $y=y_0$, where y_0 is the unique positive root of

$$\frac{\operatorname{sh} \pi y_0}{\pi y_0} = 2. \tag{8}$$

Now assume $x>0$; making $y=0$ in (6), we would obtain $(\sin \pi x)/\pi x = 2$, whereas the quotient to the left always lies between -1 and $+1$. We must consequently assume $x>0$, $y>0$, and (5) and (6) show that $\sin \pi x > 0$, $\cos \pi x > 0$, whence

$$x = 2n + \tfrac{1}{2} - \xi, \tag{9}$$

n a positive integer or zero, $0<\xi<\frac{1}{2}$. Consider first the case $n=0$, whence $0<x<\frac{1}{2}$; since $\pi x/\sin \pi x < \pi/2$ in this interval, (6) gives $\operatorname{ch} \pi y < \pi$, whence $\pi y < 1.812$, and we obtain from (7)

$$\cos \pi x > \frac{2 \times 1.812}{\operatorname{sh} 1.812} = 1.3336 > 1,$$

which is impossible.

Next, assume $n>0$; the equations (6) and (7) become

$$\cos \pi\xi \operatorname{ch} \pi y=(4n+1)\pi-2\pi\xi , \tag{10}$$

$$\sin \pi\xi \operatorname{sh} \pi y=2\pi y . \tag{11}$$

Since $\pi y/\operatorname{sh} \pi y$ is a decreasing function of y, equation (11) represents a curve in the rectangular coördinates ξ, y such that y decreases steadily from $+\infty$ to y_0 when ξ increases from 0 to $\frac{1}{2}$. On the curve (10), y is an extremum when

$$\cos^2 \pi\xi \operatorname{sh} \pi y \frac{dy}{d\xi} = [(4n+1)\pi-2\pi\xi] \sin \pi\xi-2 \cos \pi\xi=0 ,$$

and writing this equation in the form $(4n+1)\pi-2\pi\xi-2 \cot \pi\xi=0$, we see that it has a unique root ξ_0 in the interval $0<\xi<\frac{1}{2}$, since the derivative of the left hand member is $2\pi \cot^2 \pi\xi>0$. This root evidently corresponds to a minimum, since $y\to+\infty$ when $\xi\to\frac{1}{2}$ by (10), and we have

$$\sin \pi\xi_0< \tan \pi\xi_0 = \frac{2}{(4n+1)\pi-2\pi\xi_0} < \frac{1}{2n\pi} .$$

For $0<\xi<\xi_0$, the value of y obtained from (10) decreases steadily from y_1 given by $\operatorname{ch} \pi y_1=(4n+1)\pi$, to y_2, given by $\cos \pi\xi_0 \operatorname{ch} \pi y_2=(4n+1)\pi-2\pi\xi_0$, while the value of y obtained from (11) decreases from $+\infty$ to y_3, given by $\sin \pi\xi_0 \operatorname{sh} \pi y_3=2\pi y_3$. We shall now show that $y_3>y_1$; it is evidently sufficient to prove that

$$\frac{\operatorname{sh} \pi y_3}{\pi y_3} > \frac{\operatorname{sh} \pi y_1}{\pi y_1} .$$

From $e^{\pi y_1}<e^{\pi y_1}+e^{-\pi y_1}=2 \operatorname{ch} \pi y_1=(8n+2)\pi$ it follows that $\pi y_1<\log(8n+2)\pi$, and since $(\operatorname{sh} \pi y)/\pi y$ is a decreasing function,

$$\frac{\operatorname{sh} \pi y_1}{\pi y_1} < \frac{\operatorname{sh} [\log (8n+2)\pi]}{\log (8n+2)\pi} < \frac{\frac{1}{2}e^{\log (8n+2)\pi}}{\log (8n+2)\pi} = \frac{(4n+1)\pi}{\log (8n+2)\pi} .$$

On the other hand,

$$\frac{\operatorname{sh} \pi y_3}{\pi y_3} = \frac{2}{\sin \pi\xi_0} > 4n\pi ,$$

and since we have

$$4n\pi > \frac{(4n+1)\pi}{\log (8n+2)\pi}$$

for $n\geqq 1$, the inequality in question is established, and it follows that the two curves (10) and (11) do not intersect in the interval $0<\xi<\xi_0$. On the

contrary, in the interval $\xi_0<\xi<\frac{1}{2}$ the y in (10) increases steadily from y_2 to $+\infty$, while the y in (11) decreases steadily from y_3 to y_0, and since $y_3>y_1>y_2$, the two curves have a unique point of intersection in the interval considered.

Returning to (5), it is thus seen that the only zeros of this function are the following:

The two zeros $y_0 i$, $-y_0 i$;

In each of the strips $2n<\Re z<2n+\frac{1}{2}$ $(n=1, 2, 3, \cdots)$, two conjugate complex zeros;

In each of the strips $-2n-\frac{1}{2}<\Re z<-2n$ $(n=1, 2, 3, \cdots)$, two conjugate complex zeros, equal to the preceding ones multiplied by -1.

5. To find the distribution of the zeros of $\beta(1-z)$, we shall use the following theorem:

Let $f(z)$ and $g(z)$ be two functions, meromorphic inside a contour C, and holomorphic on C. When the inequality

$$|f(z)-g(z)|<|g(z)|$$

*is satisfied everywhere on C, then neither $f(z)$ nor $g(z)$ vanishes on C, and the difference between the number of zeros and the number of poles of $f(z)$ inside C equals the corresponding difference for $g(z)$.**

Let us apply this theorem to $f(z)=\beta(1-z)$ and $g(z)=(\pi/\sin\pi z)-(1/2z)$, C being the rectangle with vertices at $2n-\frac{1}{2}\pm bi$, $2n+\frac{3}{2}\pm bi$, where n is a positive integer, and b positive and very large. On the horizontal sides of the rectangle, we have $z=2n-\frac{1}{2}+x\pm bi$, $0\leqq x\leqq 2$, whence

$$\sin\pi z=-\cos\pi x\operatorname{ch}\pi b\pm i\sin\pi x\operatorname{sh}\pi b,\quad |\sin\pi z|^2=\cos^2\pi x+\operatorname{sh}^2\pi b$$

and

$$\left|\frac{\pi}{\sin\pi z}\right|\leqq\frac{\pi}{\operatorname{sh}\pi b}<2\pi e^{-\pi b}.$$

* In the case when C is a circle and $f(z)$ and $g(z)$ have no poles in its interior, this theorem is due to E. Rouché, *Mémoire sur la série de Lagrange*, Journal de l'Ecole Polytechnique, Cahier 39 (1862), pp. 193–224 (see Theorem III, p. 217). The theorem was rediscovered and generalized by A. Hurwitz, *Ueber die Nullstellen der Bessel'schen Function*, Mathematische Annalen, vol. 33 (1889), p. 246–266 (see p. 248). Incidentally, the proof is extremely simple: First, neither $f(z)$ nor $g(z)$ vanishes on C, since $|f(z)-g(z)|<|g(z)|$ yields the impossible inequalities $|f(z)|<0$ for $g(z)=0$ and $|g(z)|<|g(z)|$ for $f(z)=0$. In the identity

$$\log f(z)=\log g(z)+\log[1+(f(z)-g(z))g(z)],$$

we perform the analytic continuation of both members along the contour C, described once in the positive sense. Then $\log f(z)$ will increase by $2\pi i$ times the difference between the number of zeros and the number of poles of $f(z)$ interior to C, the increase in $\log g(z)$ will be the corresponding expression for $g(z)$, and the third logarithm does not change, since, on account of the inequality $|f(z)-g(z)|<|g(z)|$, the point $\zeta=[f(z)-g(z)]g(z)$ describes a closed path interior to the circle $|\zeta|<1$ where $\log(1+\zeta)$ is holomorphic.

Moreover, for $b>2n+2$,

$$\frac{1}{2|z|}>\frac{1}{4n-1+2x+2b}>\frac{1}{4b},\quad \frac{1}{|z|^2}=\frac{1}{(2n-\frac{1}{2}+x)^2+b^2}<\frac{1}{b^2},$$

whence, for b sufficiently large and using (4),

$$\left|\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right|\geqq\frac{1}{2|z|}-\left|\frac{\pi}{\sin\pi z}\right|>\frac{1}{4b}-2\pi e^{-\pi b},$$

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right)\right|<\frac{5}{12|z|^2}<\frac{5}{12b^2}<\frac{1}{4b}-2\pi e^{-\pi b}$$

$$<\left|\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right|.$$

On the vertical sides of the rectangle, we have $z=2m-\frac{1}{2}+yi$, $m=n$ or $n+1$ and $-b\leqq y\leqq b$, whence $\sin\pi z=-\operatorname{ch}\pi y$ and

$$\frac{\pi}{\sin\pi z}-\frac{1}{2z}=-\left(\frac{\pi}{\operatorname{ch}\pi y}+\frac{1}{4m-1+2yi}\right);$$

the real part of the second term to the right having the same sign as the first term, it follows that

$$\left|\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right|>\left|\frac{1}{4m-1+2yi}\right|=\frac{1}{|2z|}.$$

Hence we see that

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right)\right|<\frac{5}{12|z|^2}<\frac{1}{2|z|}<\left|\frac{\pi}{\sin\pi z}-\frac{1}{2z}\right|.$$

For b sufficiently large, the conditions of the theorem are this fulfilled. According to paragraph 4, the function $(\pi/\sin\pi z)-(1/2z)$ has two zeros in our rectangle for b sufficiently large, viz., those with their real part between $2n$ and $2n+\frac{1}{2}$, and on the other hand, there are the two poles $2n$ and $2n+1$. The function $\beta(1-z)$ has the same poles in the rectangle, and our theorem therefore shows that the rectangle contains exactly two zeros of $\beta(1-z)$. Letting b increase indefinitely, and replacing z by $1-z$, it is thus seen that for $n=1, 2, 3, \cdots$, each strip $-2n-\frac{1}{2}<\Re z<-2n+\frac{3}{2}$ contains two zeros of $\beta(z)$.

6. To obtain asymptotic expressions for the zeros, we begin with those of $(\pi/\sin\pi z)-(1/2z)$. As in paragraph 4, it is sufficient to consider the case $x>0$, $y>0$, and it follows from (10) that $\operatorname{ch}\pi y>4n\pi$, so that y increases

indefinitely with n, and formula (11) shows that then $\lim \sin \pi\xi = 0$, whence $\lim \cos \pi\xi = 1$. By (10), we now have

$$\lim \frac{\operatorname{ch} \pi y}{(4n+1)\pi} = 1 = \lim \frac{e^{\pi y}}{2(4n+1)\pi},$$

$$\pi y = \log [(8n+2)\pi(1+\eta)], \qquad \lim \eta = 0.$$

Substituting this value of πy in (11), it is seen that

$$\lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \pi\xi = \lim \frac{\pi\xi}{\sin \pi\xi} \cdot \lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \frac{2 \log [(8n+2)\pi(1+\eta)]}{\frac{1}{2}\left[(8n+2)\pi(1+\eta) - \frac{1}{(8n+2)\pi(1+\eta)} \right]} = 1,$$

$$\pi\xi = \frac{2 \log (8n+2)\pi}{(4n+1)\pi}(1+\xi'), \qquad \lim \xi' = 0.$$

Denoting, as usual, by $f(x) = O(g(x))$ the fact that two constants A and x_0 exist such that $|f(x)| < Ag(x)$ for $x > x_0$, we find by substituting the above values of πy and $\pi\xi$ in (10) and expanding $\cos \pi\xi$ in a power series, so that $\cos \pi\xi = 1 + O(\xi^2) = 1 + O((\log n)/n)^2$,

$$\left[1 + O\left(\left(\frac{\log n}{n} \right)^2 \right) \right] \left[(4n+1)\pi(1+\eta) - \frac{1}{4(4n+1)\pi(1+\eta)} \right] = (4n+1)\pi + O\left(\frac{\log n}{n} \right),$$

and it follows that

$$\eta = O\left(\left(\frac{\log n}{n} \right)^2 \right).$$

Finally (11) gives, taking account of the order of magnitude of η,

$$\left[\frac{2 \log (8n+2)\pi}{(4n+1)\pi} (1+\xi') + O\left(\left(\frac{\log n}{n} \right)^3 \right) \right] \left\{ (4n+1)\pi \left[1 + O\left(\left(\frac{\log n}{n} \right)^2 \right) \right] - \frac{1}{4(4n+1)\pi} \left[1 + O\left(\left(\frac{\log n}{n} \right)^2 \right) \right] \right\}$$

$$= 2 \left[\log (8n+2)\pi + O\left(\left(\frac{\log n}{n} \right)^2 \right) \right],$$

$$2 \log (8n+2)\pi \cdot \xi' = O\left(\frac{(\log n)^3}{n^2} \right),$$

whence

$$\xi' = O\left(\left(\frac{\log n}{n}\right)^2\right).$$

For the two zeros of (5) in the strip $2n < \Re z < 2n+\frac{1}{2}$, we consequently have the asymptotic expression

$$(12)\quad 2n + \frac{1}{2} - \frac{2\log(8n+2)\pi}{(4n+1)\pi^2} \pm i\,\frac{\log(8n+2)\pi}{\pi} + O\left(\left(\frac{\log n}{n}\right)^2\right).$$

7. We shall now prove that for n sufficiently large, $\beta(1-z)$ has one zero in the neighborhood of each of the zeros (12) of its principal part (5). Consider the circle

$$(13)\quad z = 2n + \frac{1}{2} - \frac{2\log(8n+2)\pi}{(4n+1)\pi^2} + i\,\frac{\log(8n+2)\pi}{\pi} + \frac{\xi+\eta i}{2n\pi},$$

$$\xi^2+\eta^2 < 1;$$

by (12), this circle contains in its interior a single zero of (5) when n is sufficiently large. On the circumference $\xi^2+\eta^2=1$, we have

$$\sin \pi z = \frac{1}{2i}\left[\frac{i}{(8n+2)\pi}\, e^{-\frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\xi i-\eta}{2n}} + i(8n+2)\pi e^{\frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\eta-\xi i}{2n}}\right]$$

$$= (4n+1)\pi\left[1 + \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\eta-\xi i}{2n} + O\left(\left(\frac{\log n}{n}\right)^2\right)\right],$$

$$\frac{\pi}{\sin \pi z} = \frac{1}{4n+1}\left[1 - \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + \frac{\xi i-\eta}{2n} + O\left(\left(\frac{\log n}{n}\right)^2\right)\right],$$

$$\frac{1}{2z} = \frac{1}{(4n+1)\left[1 + \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + O\left(\frac{\log n}{n^2}\right)\right]}$$

$$= \frac{1}{4n+1}\left[1 - \frac{2i\log(8n+2)\pi}{(4n+1)\pi} + O\left(\frac{\log n}{n^2}\right)\right],$$

whence

$$\frac{\pi}{\sin \pi z} - \frac{1}{2z} = \frac{\xi i-\eta}{2n(4n+1)} + O\left(\frac{(\log n)^2}{n^3}\right) = \frac{\xi i-\eta}{8n^2} + O\left(\frac{(\log n)^2}{n^3}\right),$$

and since $\xi^2+\eta^2=1$,

$$\left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right| = \frac{1}{8n^2}+O\left(\frac{(\log n)^2}{n^3}\right). \tag{14}$$

On the other hand, we have on the same circumference, according to (4),

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \frac{5}{12\,|z|^2} = \frac{5}{48n^2}+O\left(\frac{\log n}{n^3}\right),$$

and by comparison with (14)

$$\left|\beta(1-z)-\left(\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right)\right| < \left|\frac{\pi}{\sin \pi z}-\frac{1}{2z}\right|$$

on the circumference $\xi^2+\eta^2=1$ for n sufficiently large. Applying the theorem in paragraph 5, we see that the circle $\xi^2+\eta^2<1$ contains a single zero of $\beta(1-z)$, and the same is evidently true of the circle obtained by changing the sign of i in (13). We have thus obtained asymptotic expressions for the two zeros of $\beta(1-z)$ contained in the strip $2n-\frac{1}{2}<\Re z<2n+\frac{3}{2}$, and replacing z by $1-z$, we finally arrive at the result stated in paragraph 1.

It is clear that closer approximations to the zeros may be obtained by using, instead of (5), the general asymptotic expansion

$$\beta(z)=\frac{1}{2z}+\sum_{\nu=1}^{n}(-1)^{\nu-1}\frac{2^{2\nu}-1}{2\nu}B_\nu\cdot\frac{1}{z^{2\nu}} + \frac{1}{z^{2n}}\int_0^\infty e^{-zu}\frac{d^n}{du^n}\left(\frac{1}{1+e^{-u}}\right)du,$$

valid for $\Re z>0$, where B_ν are the Bernoulli numbers.

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OSCULATING CURVES AND SURFACES*

BY
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1. Introduction

The limit of a sequence of curves, each of which intersects a fixed curve in $n+1$ distinct points, which close down to a given point in the limit, is a curve having contact of the nth order with the fixed curve at the given point. If we substitute surface for curve in the above statement, the limiting surface need not osculate the fixed surface, no matter what value n has, unless the points satisfy certain conditions. We shall obtain such conditions on the points, our conditions being necessary and sufficient for contact of the first order, and sufficient for contact of higher order.

By taking as the sequence of curves parabolas of the nth order, we obtain theorems on the expression of the nth derivative of a function at a point as a single limit. For the surfaces, we take paraboloids of high order, and obtain such expressions for the partial derivatives. In this case, our earlier restriction, or some other, on the way in which the points close down to the limiting point is necessary.

In connection with our discussion of osculating curves, we obtain some interesting theorems on osculating conics, or curves of any given type, which follow from a generalization of Rolle's theorem on the derivative to a theorem on the vanishing of certain differential operators.

2. Osculating curves

Our first results concerning curves are more or less well known, and are presented chiefly for the sake of completeness and to orient the later results.† We begin with a fixed curve c, whose equation, in some neighborhood of the point $x=a$, $|x-a|<Q$, may be written

$$y = f(x).$$

We also consider a sequence of curves C_n, with equations

$$y = F_n(x)$$

in the same interval. Furthermore we assume that in this interval $f(x)$, as well as the $F_n(x)$, has continuous kth derivatives. In addition, let each of the curves C_n intersect c in $k+1$ points, b_i, where

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† Cf. e.g., C. Jordan, *Cours d'Analyse*, 3d edition, vol. 1, pp. 423 ff.; Goursat-Hedrick, *Mathematical Analysis*, pp. 450 ff.

$$a - Q < a - \epsilon_n < b_1 < b_2 < \cdots < b_{k+1} < a + \epsilon_n < a + Q$$

and

$$\lim_{n\to\infty} \epsilon_n = 0.$$

If, now, there is a limiting curve C with equation

$$y = F(x)$$

such that, when n becomes infinite, $F_n(x)$ and its first k derivatives approach $F(x)$ and its first k derivatives uniformly in the interval $|x-a|<Q$, then this curve C has contact of the kth order with c at the point $x=a$.

To prove this we note that the function $f(x)-F_n(x)$ vanishes $k+1$ times in the interval $|x-a|<\epsilon_n$. Also it has a continuous kth derivative there. Hence, by Rolle's theorem, this derivative must vanish at some point in the interval, say at $x=a_n$. That is,

$$f^{(k)}(a_n) - F_n^{(k)}(a_n) = 0, \qquad |a_n - a| < \epsilon_n.$$

But, by the continuity of $f^{(k)}(x)$, we may find a δ_η such that

$$|f^{(k)}(x) - f^{(k)}(a)| < \eta/3 \quad \text{if} \quad |x - a| < \delta_\eta.$$

Again, from the uniform convergence of $F_n^{(k)}(x)$ to $F^{(k)}(x)$, we may find an N such that

$$|F_n^{(k)}(x) - F^{(k)}(x)| < \eta/3 \quad \text{if} \quad n > N, \quad |x - a| < Q.$$

Finally, since $F_n^{(k)}(x)$ is continuous and converges uniformly to $F^{(k)}(x)$, this latter function is continuous, and we may find a δ_η' such that

$$|F^{(k)}(x) - F^{(k)}(a)| < \eta/3 \quad \text{if} \quad |x - a| < \delta_n'.$$

Now take $n>N$, and also so large that $\epsilon_n<\delta_\eta$, δ_η'. Since $|a_n-a|<\epsilon_n$, we may put $x=a_n$ in all the above inequalities, and combine them to get

$$|f^{(k)}(a) - F^{(k)}(a)| < \eta,$$

and hence, since η is arbitrary,

$$F^{(k)}(a) = f^{(k)}(a).$$

It is to be noted that in the above there is no restriction whatever on the way the points close down on the given point. We may formulate our result in

THEOREM I. *If a fixed curve represents a function, with a continuous kth derivative in some neighborhood of a given point on it and a sequence of curves approaching a limiting curve uniformly in this neighborhood, the uniformity applying to the first k derivatives of the representing functions, each intersects the fixed curve in $k+1$ points, which close down to the given point as we approach the limit, then the limiting curve has contact of the kth order with the fixed curve at the given point.*

3. Application to derivatives

Consider the situation of the previous section, but let us begin with the points b_i, and define the curve C_n as the parabola of the kth degree passing through these $k+1$ points. We know nothing about the limiting curve C, though later we shall show it exists. We write as the equation of C_n

$$y = F_n(x) = A_{k,n}x^k + A_{k-1,n}x^{k-1} + \cdots + A_{1,n}x + A_{0,n}.$$

The kth derivative of this function is

$$F_n^{(k)}(x) = k!\,A_{k,n}.$$

Reasoning from Rolle's theorem, as before, we find that for some point a_n this derivative equals that of $f(x)$, or

$$f^{(k)}(a_n) - k!\,A_{k,n} = 0, \qquad |a_n - a| < \epsilon_n.$$

But, by the continuity of $f^{(k)}(x)$, we may find a δ_η such that

$$|f^{(k)}(x) - f^{(k)}(a)| < \eta \quad \text{if} \quad |x - a| < \delta_\eta.$$

Accordingly, if N is so large that when $n>N$, $\epsilon_n<\delta_\eta$, since $|a_n-a|<\epsilon_n$, we may put $x=a_n$, combine the last two equations, and obtain

$$|f^{(k)}(a) - k!\,A_{k,n}| < \eta.$$

It follows from this that as n becomes infinite, $A_{k,n}$ approaches a limit, and, in fact

$$\lim_{n\to\infty} k!\,A_{k,n} = f^{(k)}(a).$$

We may easily express $A_{k,n}$ directly in terms of the $k+1$ points used to determine the parabola. For, if the curve $y=F_n(x)$ above goes through the points (x_1, y_1), (x_2, y_2), $\cdots$, (x_{k+1}, y_{k+1}), we have

$$A_{k,n} = \begin{vmatrix} y_1 & x_1^{k-1} & \cdots & x_1 & 1 \\ y_2 & x_2^{k-1} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ y_{k+1} & x_{k+1}^{k-1} & \cdots & x_{k+1} & 1 \\ \hline x_1^k & x_1^{k-1} & \cdots & x_1 & 1 \\ x_2^k & x_2^{k-1} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_{k+1}^k & x_{k+1}^{k-1} & \cdots & x_{k+1} & 1 \end{vmatrix}.$$

Noting that the denominator, and co-factors of the y_i in the numerator, are Vandermonde determinants, or products of differences, we may transform this into

$$A_{k,n} = \sum_{j=1}^{k+1} \frac{y_j}{(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_{k+1})}.$$

This establishes

THEOREM II. *If a curve represents a function* $(y=f(x))$ *with a continuous* k*th derivative in some neighborhood of a fixed point* $(x=a)$, *and* $x_1, x_2, \cdots, x_{k+1}$ *are the abscissas of* $k+1$ *variable points in this neighborhood, then, as these points close down to the fixed point,*

$$|x_i - a| < \epsilon_n \ (i = 1, 2, \cdots k+1), \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the expression

$$k! \sum_{j=1}^{k+1} \frac{f(x_j)}{(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_{k+1})}$$

approaches a limit, and this limit is $f^{(k)}(a)$.

We may note that this theorem is not vitiated if, in the process of closing down, we first let two or several of the points coincide, at a or elsewhere, provided that, as the points approach coincidence, we take the limit of the summation, which will then involve some of the derivatives of $f(x)$ and then take the limit of this. This follows from the fact that Theorem I holds if some of the points are coincident, m coincident points of intersection being interpreted to mean that the curves have contact of the $(m-1)$st order at the point.

The special case of Theorem II in which the points are equally spaced, is known.*

4. OSCULATING PARABOLAS

We shall now show that if a sequence of parabolas of the kth order having $k+1$ points of intersection with a given curve be formed, as in § 3, then, as the points close down to a fixed point, these parabolas approach the osculating parabola of the kth order at this point.

The argument of § 3 showed that the kth derivative of the approximating parabolas approached as a limit the kth derivative of the curve at the fixed point. If we consider any derivative of lower order, the qth, we may

* Cf. Ch.-J. de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 3d edition, Paris, 1914, vol. 1, p. 104; also G. Rutledge, these Transactions, vol. 26 (1924), p. 119.

prove as before the existence of a point $(x=a_{n,q})$ at which the qth derivative of the parabola and the fixed curve agree. Then, from the continuity of the qth derivative, and the fact that the points are closing down, we shall have

$$|f^{(q)}(a_{n,q}) - f^{(q)}(a)| < \eta/2, \quad n > M_q,$$

or, from the definition of $a_{n,q}$,

$$|F_n^{(q)}(a_{n,q}) - f^{(q)}(a)| < \eta/2, \quad n > M_q.$$

Since $F_n^{(k)}(x)$ is constant, and approaches $f^{(k)}(a)$ as a limit when the points close down, we may write

$$|F_n^{(k)}(x) - f^{(k)}(a)| < \epsilon', \quad n > N_1.$$

This shows that

$$|F_n^{(k)}(x)| < |f^{(k)}(a)| + \epsilon' = G_1, \quad n > N_1.$$

Integrating this, we obtain

$$|F_n^{(k-1)}(x) - F_n^{(k-1)}(a)| < G_1|x - a|, \quad n > N_1.$$

Since $|a-a_{n,q}|<\epsilon_n$, if we take N_2 greater than N_1, and also so large that $\epsilon_{N_2}<\eta/2G_1$, we may take $x=a_{n,k-1}$, and get

$$|F_n^{(k-1)}(a_{n,k-1}) - F_n^{(k-1)}(a)| < \eta/2, \quad n > N_2.$$

On combining this with the earlier inequality for $f^{(q)}(a)$, $q=k-1$, there results

$$|F_n^{(k-1)}(a) - f^{(k-1)}(a)| < \eta, \quad n > N_2, \quad M_{k-1}.$$

This shows that the $(k-1)$st derivatives of the approximating parabolas at the point $x=a$ approach as a limit the $(k-1)$st derivative of the fixed curve at this point. From this fact, and the upper bound for the kth derivative $(G_1, n>N_2)$, we may obtain an upper bound for the $(k-1)$st derivative $(G_2, n>N_3, |x-a|<\epsilon_n)$. Then an argument similar to the above proves that the $(k-2)$d derivative of the approximating parabola at $x=a$ approaches $f^{(k-2)}(a)$ as a limit. Proceeding in this way, using at each stage the constancy of the kth derivative, and the limit, at $x=a$, of the intermediate derivatives, we may prove a similar result for all the derivatives of the approximating parabolas, and, finally, for their ordinates, at $x=a$.

Since the coefficients of a parabola of the kth degree are uniquely determined by, and in fact, continuous functions of, the values of an ordinate and all the derivatives at one point, we see that the coefficients of the approximating parabolas approach those of the parabola of the kth order osculating the fixed curve at $x=a$ to this order. We have thus proved

THEOREM III. *If a curve represents a function with a continuous kth derivative in some neighborhood of a fixed point* $(x=a)$, *and a parabola of the kth order is passed through* $k+1$ *points* $(x=x_i,\ i=1, 2, \cdots, k+1)$ *on the curve, then, as these points close down to the fixed point,*

$$| x_i - a | < \epsilon_n, \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the parabola will approach the parabola of the kth order, which osculates the curve to this order at the fixed point, in the sense that its coefficients will approach those of the limiting curve, and hence, its ordinates and derivatives will approach those of the limiting curve uniformly in any finite neighborhood of $x=a$.

5. A GENERALIZATION OF ROLLE'S THEOREM

In the derivation of the preceding results, the use of Rolle's theorem was essential. To extend these results, we require a generalization of Rolle's theorem, to which we now proceed. The form of statement of Rolle's theorem here applied stated that, if a parabola of the kth degree (solution of the differential equation $d^{k+1}y/dx^{k+1}=0$) had $k+1$ points in common with a curve possessing a continuous kth derivative, then, at some intermediate point, the value of the kth derivative was the same for the parabola and the curve (a first integral of the differential equation, $d^ky/dx^k=c$, was satisfied at some intermediate point on the curve). A natural generalization of this is the following:

If $L_{k+1}(y^{(k+1)}, y^{(k)}, \cdots, y', y, x)=0$ *is a differential equation whose left member satisfies certain continuity and solvability conditions, having a first integral* $L_k(y^{(k)}, \cdots, y', y, x, c_0)=0$, *which leads to the general solution* $L(y, x, c_0, c_1, \cdots, c_k)=0$, *then if the c's are determined so that* $L=0$ *is satisfied at* $k+1$ *points of a given curve with continuous kth derivative in a neighborhood including these points, then* $L_k=0$ *is satisfied at some intermediate point on the curve.*

The conditions of continuity and solvability referred to are such as guarantee that, at each stage of the process to be used, the indicated formal solutions for implicit functions exist, and lead to functions with continuous derivatives. Our process is as follows:

We solve $L(y, x, c_0, c_1, \cdots, c_k)=0$ for c_k, obtaining

$$M(y, x, c_0, c_1, \cdots, c_{k-1}) - c_k = 0.$$

If the left member be regarded as a function of x through the equation of

the curve, it vanishes at the $k+1$ points of intersection. Accordingly its derivative vanishes at k intermediate points. If

$$M' = L_1(y',\ y,\ x,\ c_0,\ c_1,\ \cdots,\ c_{k-1}) = 0$$

be solved for c_{k-1}, giving

$$M_1(y',\ y,\ x,\ c_0,\ c_1,\ \cdots,\ c_{k-2}) - c_{k-1} = 0,$$

the left member, regarded as a function of x through the equation of the curve, vanishes at the k intermediate points previously found. Accordingly its derivative vanishes at $k-1$ intermediate points.

Continuing in this way, we finally find that

$$M'_{k-1} = L_k(y^{(k)},\ \cdots,\ y',\ y,\ x,\ c_0) = 0$$

is satisfied by some intermediate point on the curve.

In general, it is difficult to insure in advance that each time we solve an equation $L_q = 0$ for one of the constants, the resulting function will be continuous and differentiable. In one case of fairly wide applicability however, we can predict these properties. This is the case of equations whose general solution $L = 0$ involves the constants c *linearly*. We note in passing that as the functions multiplying the c's involve both x and y, we are not restricted to linear differential operators. We may write

$$L(y,\ x,\ c_0,\ c_1,\ \cdots,\ c_k) = \sum_{i=0}^{k} c_i z_i(x,\ y) + z_{k+1}(x,\ y).$$

If we assume that the z's possess continuous partial derivatives up to the $(k+1)$st order, since our equations are linear in the c's and remain so after we solve for one of them and differentiate, the M's will all be continuous and differentiable unless one of the denominators which occurs in the calculation vanishes. It is easily found by direct computation that these denominators are, essentially, the Wronskians of the z's, considered as functions of x (both directly and through y):

$$W_k,\ W_{k,k-1},\ \cdots,\ W_{k,k-1,\cdots,1,0},$$

or, explicitly,

$$z_k,\ \begin{vmatrix} z_k & z_{k-1} \\ z'_k & z'_{k-1} \end{vmatrix},\ \cdots,\ \begin{vmatrix} z_k & z_{k-1} & \cdots & z_0 \\ z'_k & z'_{k-1} & \cdots & z'_0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ z_k^{(k)} & z_{k-1}^{(k)} & \cdots & z_0^{(k)} \end{vmatrix}.$$

The last one is included so that we may solve L_k for c_0, as we shall wish to do presently.

Thus, when restricted to the case of integrals involving the constants linearly, the statement made above may be made more precisely as follows:

THEOREM IV. *If, in some neighborhood, a given curve has a continuous kth derivative, and the functions* $z_i(x, y)$ $(i=0, 1, \cdots, k)$ *are such that none of the Wronskians* W_k, $W_{k,k-1}, \cdots, W_{k,k-1,\cdots,1,0}$ *taken along the curve vanish in the neighborhood (these* $z_i(x, y)$, *as well as* $z_{k+1}(x, y)$ *having continuous partial derivatives of the first k orders), and if the constants in* $L=\sum_{i=0}^{k} c_i z_i(x, y)+z_{k+1}(x, y)$ *are so determined that* $L=0$ *at* $k+1$ *points on the given curve, in the given neighborhood the equation* $L_k(y^{(k)}, \cdots, y', y, x)=c_0$, *obtained from* $L=0$ *by elimination of the constants, is satisfied at some intermediate point of the curve.*

In addition to the linear case, there are certain other special cases where the process described above may be carried through. For example, if we take the differential equation of all the circles in the plane, and the radius as one of the constants (c_0), we find that the solutions are continuous if we keep in a neighborhood less than the diameter of the circle. As this may always be done if the arc is less than a semicircle, by a suitable choice of axes, we have the following result:

If a curve with continuous curvature has three points in common with a semicircle, it has the same curvature as the semicircle at some intermediate point.

6. OSCULATING CONICS, OR CURVES OF GIVEN TYPE

We are now in a position to extend the treatment of osculating parabolas to curves of other types. We begin with osculating conics. We assume that the fundamental curve has, in some neighborhood of a given point, a representing function with a continuous fourth derivative. Also that at the given point, its osculating conic is uniquely determined. Analytically, this requires that the matrix formed by dropping the last row of the Wronskian of the quantities $1, x, y, x^2, xy, y^2$ should be of rank five. The rank will fall if and only if $y''=y'''=0$, in which case there is a point at which the tangent has contact of the third order with the curve. Hence this tangent and any other line through the point will serve as an osculating conic. Excluding this exceptional case, there is some determinant of the fifth order in the matrix in question which does not vanish. Thus we may so number the quantities $1, x, y, x^2, xy, y^2$ as $z_i(x\ y)$ $(i=0, 1, 2, \cdots, 5)$ that the Wronskians $W_4, W_{43}, \cdots, W_{43210}$ are all different from zero at the given point, and, since they are continuous, in some neighborhood of the given point.

Now consider a sequence of conics, each of which has five points in common with the fundamental curve in the neighborhood of the given

point, the points closing down to the given point as we run out in the sequence. All the conditions of Theorem IV are satisfied, and we infer that $L_4(y^{(4)}, \cdots, y', y, x) = c_0$ is satisfied at some intermediate point of the curve. Hence, as the points close down, this expression approaches a limit, and this limit is the value of c_0 for the osculating conic at the given point. We now use an argument similar to that of § 4, proceeding from the boundedness of L_4 in some neighborhood of the given point, by integration, to an inequality on L_3, and then by a second application of Theorem IV to the proof that L_3 approaches as a limit c_1, its value for the osculating conic. A repetition of the process shows that all the c_i approach limits, and the curve approaches a limiting curve, the osculating conic. Furthermore, since each L is linear in the highest derivative it contains, and may be solved for it, the approach to the limiting curve is in the sense previously used, i. e., uniform approach of all the derivatives used. This gives

THEOREM V. *If a curve represents a function with a continuous fourth derivative in some neighborhood of a fixed point* $(x=a)$ *and at this point has a uniquely determined osculating conic, and a conic is passed through five points on the curve* $(x=x_i, i=1, 2, \cdots, 5)$ *then, as these points close down to the fixed point,*

$$|x_i - a| < \epsilon_n, \quad \lim_{n\to\infty} \epsilon_n = 0,$$

the conic will approach the osculating conic at the fixed point, in the sense that its coefficients will approach those of the limiting curve and its ordinates and their first four derivatives will approach those of the limiting curve uniformly in any finite neighborhood of $x=a$.

The proof of Theorem V is applicable, mutatis mutandis, to any given type of osculating curve, provided that the constants enter linearly, and the functions entering into it are sufficiently differentiable. In particular, we note the case of the circles through three points approaching the osculating circle. Here there is no exceptional case, as the continuity of the second derivative insures the uniqueness of the osculating circle.

7. TANGENT PLANES TO SURFACES

We shall begin our discussion of osculating surfaces with the simple case of a tangent plane, and our first problem is to determine under what conditions three points on a given surface, which close down to a fixed point, determine the tangent plane. Evidently some condition is necessary, since if the points approached the fixed point along a plane section of the surface, they would determine this section. The facts here are expressed in

THEOREM VI. *If P is a fixed point on a surface whose equation is $z=f(x, y)$, continuous and possessing continuous first partial derivatives in some neighborhood of P, and A_n, B_n, C_n are three sequences of points closing down on P, in such a way that the triangle $A_nB_nC_n$ always has at least one angle of its xy projection simultaneously greater than k and less than $\pi-k$, the plane through $A_nB_nC_n$ approaches the tangent plane to the surface at P. This last limit is approached in the sense that the distance of any point in this plane from the point in the tangent plane with the same x, y approaches zero uniformly in any finite region, and also the vertical slope of any line in this plane approaches that of the line in the tangent plane with the same xy projection.*

Since the points for a given index n may be interchanged, we may assume that in the triangle ABC, one in the sequence, which projects in the xy plane into A_1, B_1, C_1, the angle A_1 is always between k and $\pi-k$. Let A_1B_1 and A_1C_1 make angles s and t respectively with the x axis. Let R be the angle of an arbitrary line in the xy plane with the x axis. Let s_2, t_2 and R_2 be the vertical slopes of the lines in the tangent plane to the surface projecting into lines parallel to A_1B_1, A_1C_1 and the R line respectively, and let s_3, t_3, and R_3 be the corresponding slopes for lines in the plane ABC.

For a triangle in the xy plane with sides parallel to A_1B_1, A_1C_1 and the R line, if we designate the vectors forming the sides by V_s, V_t and V_R, and take $|V_R|=1$, we shall have

$$|V_s| = |\sin(t-R)/\sin(t-s)|\,; \quad |V_t| = |\sin(s-R)/\sin(s-t)|.$$

Consequently, we shall have

$$V_R = \frac{\sin(t-R)}{\sin(t-s)}\frac{V_s}{|V_s|} + \frac{\sin(s-R)}{\sin(s-t)}\frac{V_t}{|V_t|}.$$

From this we derive

$$R_2 = \frac{\sin(t-R)}{\sin(t-s)}s_2 + \frac{\sin(s-R)}{\sin(s-t)}t_2,$$

and

$$R_3 = \frac{\sin(t-R)}{\sin(t-s)}s_3 + \frac{\sin(s-R)}{\sin(s-t)}t_3.$$

But we have

$$s_2 = \cos s(\partial z/\partial x) + \sin s(\partial z/\partial y),$$

and

$$t_2 = \cos t(\partial z/\partial x) + \sin t(\partial z/\partial y),$$

where the partial derivatives are calculated from $z=f(x, y)$ for the point P. Furthermore, by Rolle's theorem, the slope of AB must equal the vertical

slope of the curve on the surface with the same xy projection at some point between A and B, say P_s, and similarly for the slope of AC and some point P_t. Denoting partial derivatives at these points with appropriate subscripts, we have

$$s_3 = \cos s(\partial z/\partial x)_s + \sin s(\partial z/\partial y)_s$$

and

$$t_3 = \cos t(\partial z/\partial x)_t + \sin t(\partial z/\partial y)_t.$$

Since the partial derivatives are continuous in some neighborhood of P, we may find a neighborhood such that in it the oscillations of $\partial z/\partial x$ and $\partial z/\partial y$ are each less than $(\eta/4)\sin k$. Since the points $A_nB_nC_n$ were closing down on P, for n sufficiently great they will lie in this neighborhood. Accordingly, if the n we started with was big enough, then both P_s and P_t will lie in it, and we shall have

$$|s_2 - s_3| < (\eta/2)\sin k\,; \quad |t_2 - t_3| < (\eta/2)\sin k.$$

But, since the angle A_1 was between k and $\pi - k$, we have

$$|\sin(t - s)| = |\sin(s - t)| > \sin k.$$

From this and the equations for R_2 and R_3 we find

$$|R_2 - R_3| < \eta,$$

which proves the part of the conclusion dealing with vertical slopes. To get from this to distances, we need merely note that in any region of finite size the distance from a point to the point P, projected in the xy plane, will be bounded, say less than G. Furthermore, since the equation of the surface involves a continuous function, if the distance of a point from P, projected on the xy plane, is sufficiently small, its vertical distance from the tangent plane will also be small, say less than $\delta/2$. Let us now take our points so so far out in the sequence that each of the projected distances A_1P_1, B_1P_1, C_1P_1 is within this limit, less than G, and also that $|R_2 - R_3| < \delta/4G$, which we may do by the inequality above. Then, if Q_2 and Q_3 are two points in the tangent plane, and plane ABC respectively, with the same xy projection Q_1, lying inside the G domain, we shall have

$$\begin{aligned} |Q_2Q_3| &\leqq |A_2A| + |R_2 - R_3|\,|A_1Q_1| \\ &< \delta/2 + (\delta/4G)\,2G = \delta. \end{aligned}$$

This completes the proof of Theorem VI.

8. Application to first partial derivatives

We may interpret the result of Theorem VI analytically, by noting that under the hypothesis there given, the equation of the plane ABC

will approach the equation of the tangent plane, in the sense of approach of the ratios of the coefficients. Or we may apply the theorem to obtain expressions approaching the partial derivatives. Here it is convenient to start with the projections of the points in the xy plane. The expressions are obtained by finding the equation of the plane through the three points on the surface in the form

$$z = Ax + By + C,$$

and comparing it with the equation of the tangent plane. The result is

THEOREM VII. *If a surface represents a function $z=f(x, y)$ with continuous first partial derivatives in some neighborhood of a fixed point (a, b, c), and (x_1, y_1), (x_2, y_2), (x_3, y_3) are three points closing down to the fixed point (a, b):*

$$|x_i - a| < \epsilon_n, \quad |y_i - b| < \epsilon_n \ (i = 1, 2, 3); \quad \lim_{n\to\infty} \epsilon_n = 0,$$

in such a way that their triangle always has at least one angle between k and $\pi - k$, then, as they close down, the expressions

$$\frac{\begin{vmatrix} f(x_1, y_1) & y_1 & 1 \\ f(x_2, y_2) & y_2 & 1 \\ f(x_3, y_3) & y_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad \frac{\begin{vmatrix} f(x_1, y_1) & x_1 & 1 \\ f(x_2, y_2) & x_2 & 1 \\ f(x_3, y_3) & x_3 & 1 \end{vmatrix}}{\begin{vmatrix} y_1 & x_1 & 1 \\ y_2 & x_2 & 1 \\ y_3 & x_3 & 1 \end{vmatrix}}$$

approach limits, and these limits are $\partial z(a, b)/\partial x$ and $\partial z(a, b)/\partial y$ respectively.

9. OSCULATING PARABOLOIDS

If $(n+1)(n+2)/2$ points are selected on a surface, one and only one paraboloid of the nth or lower degree may be passed through them. We wish to investigate certain sufficient conditions on the points, under which, as the points close down to a fixed point, the sequence of paraboloids will approach the osculating paraboloid at this point. We shall assume that the points are so selected that their projections in the xy plane may be joined by $n+1$ straight lines, in such a way that each straight line contains $n+1$ of the points, and furthermore that the smallest angle between any pair of the lines is greater than some positive constant k, k being selected for the entire sequence. This configuration is obviously possible, since we have merely to add to the n $(n+1)/2$ points of intersection of $n+1$ straight lines an additional point on each line. We shall in the future briefly refer to this situation by saying the points are on a *non-degenerating configuration.*

Along one of the $n+1$ lines just mentioned, the ith, the ordinate of the given surface, z, will be a function of the arc length on the straight line, s_i, and we shall have

$$\begin{aligned} d^n z/ds_i^n &= ((dx/ds_i)(\partial/\partial x) + (dy/ds_i)(\partial/\partial y))^n z \\ &= \sum_{j+k=n} C_{jk}^n (\partial^n z/\partial x^j \partial y^k)(dx/ds_i)^j (dy/ds_i)^k \\ &= \sum_{j+k=n} Q_{jk}^i \partial^n z/\partial x^j \partial y^k, \end{aligned}$$

where the Q's are constants depending on the slope of the ith line.

Now consider the curves on the paraboloid and on the surface which project into the straight lines just mentioned. As the two such curves which project into a given line have $n+1$ points in common, by applying Rolle's theorem to z as a function of s_i along one of them we see that there is a point, P_i, on each at which $d^n z/ds_i^n$ is the same for both of them. Again, provided the partial derivatives of the nth order for the original surface are continuous, which would insure the amount of continuity necessary for Rolle's theorem, when the variable points A_i are sufficiently close to P, i. e., when the process of closing down has been carried sufficiently far, these partial derivatives at P_i will differ from those at P by an amount less than

$$\eta' = \frac{\eta \sin {}^{(n+1)/2}k}{2^n (n+1)^{(n+1)/2}}.$$

Applying these remarks to the equation for $d^n z/ds_i^n$ given above, we see that it may be written

$$\begin{aligned} (d^n z/\, ds_i^n)_{P_i} &= \sum_{j+k=n} Q_{jk}^i (\partial^n z/\partial x^j \partial y^k)_{P_i} \\ &= \sum_{j+k=n} Q_{jk}^i (\partial^n z/\partial x^j \partial y^k)_P - \eta' \theta_i 2^n \quad (|\theta_i| \leqq 1), \end{aligned}$$

since the sum of the binomial coefficients C_{jk}^n is 2^n, and the quantities dx/ds_i and dy/ds_i are each less than unity.

Now write the equation of the paraboloid in the form

$$z = \sum_{j+k=n} p_{jk} \frac{x^j}{j!} \frac{y^k}{k!} + \sum_{j+k=n-1} p'_{jk} \frac{x^j}{j!} \frac{y^k}{k!} + \cdots.$$

The partial derivatives of the nth order for it will be constant, and equal to p_{jk}. Hence the earlier equation, applied to the paraboloid, gives

$$(d^n z/\, ds_i^n)_{P_i} = \sum_{j+k=n} Q_{jk}^i p_{jk}.$$

Hence, if we put

$$E_{jk} = (\partial^n z/\partial x^j \partial y^k)_P - p_{jk},$$

the departure of p_{jk} from the corresponding derivative for the surface, we find that these errors must satisfy the equations

$$\theta_i \eta' 2^n = \sum_{j+k=n} Q^i_{jk} E_{jk}.$$

Solving this system, we find

$$E_{jk} = \frac{|Q^i_{0,n} \cdots \theta_i \eta' 2^n \cdots Q^i_{n,0}|}{|Q^i_{0,n} \cdots Q^i_{jk} \cdots Q^i_{n,0}|} = \frac{N_j}{D},$$

where i varies from 1 to $n+1$ to give the successive rows of the determinants, and $k = n-j$.

If we denote the product of the binomial coefficients by C, and apply Hadamard's theorem on the maximum value of a determinant to the numerator, we have

$$|N_j| \leqq C\eta' \left(\frac{2^n}{C^n_{jk}}\right) (n+1)^{(n+1)/2}.$$

The denominator is

$$|D| = C\,|(dx/ds_i)^j (dy/ds_i)^k|$$

$$= C \prod ((dx/ds_p)(dy/ds_q) - (dx/ds_q)(dy/ds_p)),$$

since on factoring out $(dx/ds_i)^n$ it becomes a Vandermonde determinant. Denoting the inclination of the ith line to the x axis by t_i, this becomes

$$|D| = |C \prod \sin (t_p - t_q)| > C \sin^{(n+1)/2} k.$$

Consequently we have

$$|E_{jk}| \leqq \eta' 2^n (n+1)^{(n+1)/2} / \sin^{(n+1)/2} k = \eta.$$

As η approaches zero when the points close down to P, we see that the coefficients of the terms of highest degree in the equation of the paraboloid approach the partial derivatives of the surface at P.

We may carry out a similar discussion for the partial derivatives of lower degree. For, we have

$$d^{n-1}z/ds_i^{n-1} = \left((dx/ds_i)(\partial/\partial x) + (dy/ds_i)(\partial/\partial y)\right)^{n-1} z$$

$$= \sum_{j+k=n-1} Q'^i_{jk} (\partial^{n-1} z/\partial x^j \partial y^k).$$

As before, when the closing down has been carried sufficiently far, from considering the curve on the fixed surface, we shall have

$$(d^{n-1}z/ds_i^{n-1})_{P_i} = \sum_{j+k=n-1} Q'^{i}_{jk}(\partial^{n-1}z/\partial x^j\partial y^k)_{Psurf} - \theta_i\eta'2^{n-1}.$$

For the paraboloid, after a certain point in the closing down, the nth partial derivatives will be close to their limits, and therefore bounded. Hence, in a sufficiently small neighborhood of the point P, the $(n-1)$st partial derivatives will be within η' of their values at P on the paraboloid. Thus, by considering the corresponding curve on the paraboloid, we shall have the equation

$$(d^{n-1}z/ds_i^{n-1})_{P_i} = \sum_{j+k=n-1} Q'^{i}_{jk}(\partial^{n-1}z/\partial x^j\partial y^k)_{Pparab} - \theta_i'\eta'2^{n-1}.$$

Accordingly, we may put

$$E'_{jk} = (\partial^{n-1}z/\partial x^j\partial y^k)_{Psurf} - (\partial^{n-1}z/\partial x^j\partial y^k)_{Pparab},$$

and obtain the equations

$$2^{n-1}(\theta_i - \theta_i')\eta' = 2^n\theta_i''\eta' = \sum_{j+k=n-1} Q'^{i}_{jk}E'_{jk}.$$

Precisely as before, we may show from these equations that the E'_{jk} approach zero when we close down on the point P.

Proceeding in this way step by step, we may work down to the derivatives of lower order, and finally to the value of z itself. As the coefficients of a paraboloid of the nth degree are uniquely determined by, and in fact, continuous functions of, the values of an ordinate and all the partial derivatives at one point, we see that the coefficients of the approximating paraboloids apapproach those of the osculating paraboloid of the nth order to the surface at P. This proves

THEOREM VIII. *If a surface represents a function with continuous nth partial derivatives in some neighborhood of a fixed point P, and a paraboloid of the nth degree is passed through $(n+1)(n+2)/2$ points A_i on the surface, whose projections on the xy plane form a non-degenerating configuration, then, as these points close down to P,*

$$|A_i - P| < \epsilon_n, \lim_{n\to\infty} \epsilon_n = 0,$$

the paraboloid will approach the paraboloid of the nth degree which osculates the surface at P to this order, in the sense that its coefficients will approach those of the limiting surface, and hence its ordinates and derivatives will approach those of the limiting surface uniformly in any finite neighborhood of P.

10. Application to partial derivatives

The relation of the coefficients of highest degree in the equation of a paraboloid to the partial derivatives of corresponding order enables us to go directly from Theorem VIII to the construction of expressions involving the coördinates of points on a surface which approach partial derivatives of any order for this surface at a point, when the points close down to this point. We may state

THEOREM IX. *If a surface represents a function $z=f(x, y)$ with continuous nth partial derivatives in some neighborhood of a fixed point (a, b, c) and (x_i, y_i) are $(n+1)(n+2)/2$ points forming a non-degenerating configuration closing down to the fixed point (a, b):*

$$|x_i - a| < \epsilon_n, \quad |y_i - b| < \epsilon_n \, (i = 1,2, \cdots, (n+1)(n+2)/2 \, ; \lim_{n\to\infty} \epsilon_n = 0),$$

then, as they close down, the expression

$$j!\,k! \frac{\begin{vmatrix} x_i^n & \cdots & f(x_i y_i) & \cdots & x_i & y_i & 1 \end{vmatrix}}{\begin{vmatrix} x_i^n & \cdots & x_i^j y_i^k & \cdots & x_i & y_i & 1 \end{vmatrix}},$$

i varying from 1 to $(n+1)(n+2)/2$ to give the successive rows of the determinants, approaches a limit, and this limit is $\partial^n z(a, b)/\partial x^j \partial y^k$.

11. Osculating surfaces

We shall now consider a theorem for surfaces analogous to Theorem I for curves. We here consider a sequence of surfaces approaching a limiting surface uniformly, in the sense of approach of partial derivatives up to the nth order, each of which intersects a fixed surface in $(n+1)(n+2)/2$ points. We assume that the projections of these points in the xy plane form a non-degenerating configuration and that they are closing down to a fixed point P. We wish to show that the limiting surface osculates the fixed surface at P.

We obtain a sequence of paraboloids by passing one of the nth degree through each set of points. By Theorem VIII, these paraboloids approach the osculating paraboloid to the fixed surface at P. From the uniform continuity of the partial derivatives of this limiting paraboloid, we infer that their oscillation will be small in a sufficiently small neighborhood of P. Hence, from the uniform approach to this limit, we see that when we have closed down sufficiently the partial derivatives of the approximating paraboloids will, in a restricted neighborhood of P, differ slightly from those of the limiting paraboloid, or of the fixed surface, at P. Similarly, from the

uniform continuity of the partial derivatives of the limiting surface, and the uniform approach to it, we see that in a restricted neighborhood of P the partial derivatives of the approximating surfaces will differ slightly from those of the limiting surface. We may use these facts, and Rolle's theorem, to set up linear equations for the differences

$$E_{ij} = (\partial^n z/\partial x^i \partial y^k)_P^{parab} - (\partial^n z/\partial x^i \partial y^k)_P^{lim\ surf}$$

of the form

$$\theta_{i\eta} = \sum_{i+k=n} Q_{ik}^{i} E_{ik},$$

where η approaches zero as we close down to the limiting point. From these we infer, as in § 9, that the E_{ij} approach zero, and thence step down to the lower derivatives. Thus we obtain

THEOREM X. *If a sequence of surfaces representing functions with continuous nth partial derivatives approach a limiting surface uniformly, the uniformity applying to the partial derivatives of order n or lower, and these surfaces each intersect a given surface of the same type in* $(n+1)(n+2)/2$ *points, whose projections on the xy plane form a non-degenerating configuration, and which close down to a fixed point as we go out in the sequence, then the limiting surface osculates the fixed surface at the fixed point to the nth order.*

If we have two sequences of surfaces, each pair intersecting in a set of points whose projections form a non-degenerating configuration, approaching limiting surfaces, we may derive a similar result for these limiting surfaces. For, on forming the paraboloids through the sets of points, we obtain a limiting paraboloid which osculates both limiting surfaces, and accordingly they osculate each other. This leads to

THEOREM XI. *If two sequences of surfaces representing functions with continuous nth partial derivatives approach limiting surfaces uniformly, the uniformity applying to the partial derivatives of order n or lower, and each corresponding pair of surfaces, one from each sequence, intersect in* $(n+1)\cdot(n+2)/2$ *points whose projections on the xy plane form a non-degenerating configuration, and which close down to a fixed point as we go out in the sequences, then the two limiting surfaces osculate each other at the fixed point to the nth order.*

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ON LAPLACE'S INTEGRAL EQUATIONS*

BY

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The equation with which this paper is concerned is of the form

$$(\star) \qquad \int_{(C)} e^{-xz}F(x)dx = f(z),$$

which is known in the literature as Laplace's integral equation. The contour (C) and the function $f(z)$ are supposed given and $F(x)$ is to be found.

In the case when the contour (C) consists of the positive part of the axis of reals, the solution of the equation $(\star)$ was given by H. Poincaré† and H. Hamburger.‡ Each of these authors considers $F(x)$ as a function of the real variable x.

When the contour (C) consists of the entire axis of reals, a simple substitution reduces $(\star)$ to the form studied by Riemann§ and H. Mellin.‖ In the present paper we discuss the equation $(\star)$ in the case of Poincaré, extending the solution $F(x)$ to complex values of x. A certain relation of reciprocity between the functions $f(z)$ and $F(x)$ is thereby revealed.

1. Poincaré obtained the solution of the equation

$$(1) \qquad f(z) = \int_0^\infty e^{-xz}F(x)dx$$

in form of a definite integral

$$(2) \qquad F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta.$$

This solution can be easily verified assuming certain hypotheses concerning the function $f(z)$ and the contour (D).¶ Setting

$$x = u + iv$$

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* Presented to the Society, May 1, 1926; received by the editors in November, 1924.

† *Sur la théorie des quanta*, Journal de Physique, ser. 5, vol. 2 (1912).

‡ *Ueber eine Riemann'sche Formel* . . . , Mathematische Zeitschrift, vol. 6 (1920), pp. 6–9.

§ *Ueber die Anzahl der Primzahlen* . . . , Werke, 1876, p. 140.

‖ *Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen*, Mathematische Annalen, vol. 68 (1910), pp. 318–324.

¶ The integration over (D) is taken from $\zeta = \lambda - i\infty$ to $\zeta = \lambda + i\infty$. If the integral $\int_{(D)}$ fails to exist, but Cauchy's principal value exists, let it be denoted by the same symbol $\int_{(D)}$, and no further modifications are necessary.

we make the following assumptions:

(A_1) The function $f(z)$ is analytic on the half-plane

$$u > \lambda_0, \tag{3}$$

where λ_0 is a non-negative constant determined by $f(z)$. Further, if λ denotes any number $>\lambda_0$, then $f(z)$ approaches zero, uniformly for $u \geqq \lambda, |z| \to \infty$.

(A_2) If (D) denotes the straight line $u=\lambda$, the integrals

$$\frac{1}{2\pi i}\int_{(D)} e^{\xi z} f(\zeta)\,d\zeta\,, \quad \frac{1}{2\pi i}\int_0^\infty e^{-x\xi}\,d\xi \int_{(D)} e^{\xi\xi} f(\zeta)\,d\zeta$$

exist and, in the latter, the order of integration can be interchanged.

THEOREM 1. *Under the conditions* (A_1) *and* (A_2) *the function* $F(x)$ *given by* (2) *is a solution of* (1) *and does not depend on* λ.

We have

$$\int_0^\infty e^{-x\xi} F(\xi)\,d\xi = \frac{1}{2\pi i}\int_0^\infty e^{-x\xi}\,d\xi \int_{(D)} e^{\xi\xi} f(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_{(D)} f(\zeta)\,d\zeta \int_0^\infty e^{\xi(\zeta - x)}\,d\xi.$$

If z is any point in the region $u>\lambda_0$, we can always find $\lambda>\lambda_0$ such that $u>\lambda>\lambda_0$, and then

$$\int_0^\infty e^{\xi(\zeta - z)}\,d\xi = -\frac{1}{\zeta - z}$$

and

$$\int_0^\infty e^{-x\xi} F(\xi)\,d\xi = -\frac{1}{2\pi i}\int_{(D)} \frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{4}$$

The point z is situated to the right of (D) and by virtue of (A_1) the integral

$$\int \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

taken round the right half of the circle $|\zeta - \lambda| = R$, approaches 0 as $R \to \infty$. A simple application of the Cauchy integral theorem shows then that the right hand member of (4) equals $f(z)$. A similar argument shows that the value of the integral (2) remains unchanged under any parallel displacement of (D) in the region (3), in other words that the expression (2) for $F(x)$ does not depend on λ. A special case of the condition (A_2) is the following:

(A_3) The integral

$$\int_{(D)} f(\zeta)\,d\zeta \tag{5}$$

is absolutely convergent.

In this case the integral

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta z} f(\zeta)\, d\zeta \qquad (x \geqq 0)$$

exists, and the same is true of the double integral

$$\int_0^\infty \int_{(D)} \left| e^{\xi(\zeta - s)} f(\zeta) \right| d\xi \; | \, d\zeta \, | = \int_0^\infty \int_{-\infty}^{\infty} e^{-\xi(u-\lambda)} \left| f(\lambda + i\tau) \right| d\xi d\tau,$$

which assures the existence of the repeated integrals of the condition (A_2), and also the legitimacy of interchanging of the order of integration. It should be noted that the condition (A_2) is more general than the condition (A_3). For instance, the equation

$$\int_0^\infty e^{-s\xi} F(\xi)\, d\xi = z^p \qquad (\text{Re } p < 0)$$

admits of a solution

$$F(x) = \frac{1}{\Gamma(\lambda - p)}\, x^{-p-1} = \frac{1}{2\pi i}\int_{(D)} e^{\zeta z} \zeta^{-p}\, d\zeta,$$

whereas it is readily found that the function

$$f(Z) = z^p \qquad (\text{Re } p > -1)$$

satisfies both conditions (A_1) and (A_2) but not (A_3). In virtue of this example it is clear that the condition (A_3) may be replaced by the more general

(A_4) The function $f(z)$ is of the form

$$f(z) = \sum_{s=1}^{m} c_s z^{p_s} + \varphi(z),$$

where c_s are arbitrary constants, the exponents p_s satisfy the conditions $\text{Re } p_s < 0 (s = 1, 2, \cdots, n)$ and the function $\varphi(z)$ satisfies the conditions (A_1) and (A_3).

2. Thus far the solution $F(x)$ of the equation (1) which is given by the formula (2) is determined only for real positive values of x. For complex values of x the integral (2) may even become divergent. Let us now consider $F(x)$ as the given function and define $f(z)$ by the formula (1). We set

$$x = \xi + i\eta$$

and suppose that

(B_1) The function $F(x)$ is analytic in the strip

$$\xi \geqq 0\,; \qquad |\, \eta \,| < \alpha_0, \tag{6}$$

where α_0 is a positive constant determined by $F(x)$. Further, there exists a non-negative constant λ_0, depending only on α_0, such that for every pair of numbers λ, α $(\lambda>\lambda_0;\ 0<\alpha<\alpha_0)$ the product

$$e^{-\lambda z}F(x) \tag{7}$$

approaches zero uniformly as $|x|\to\infty$ in the strip $\xi\geqq 0$, $|\eta|\leqq\alpha$.

Consider now the function $f(z)$ which is defined by

$$f(z)=\int_0^\infty e^{-zt}F(\xi)d\xi. \tag{8}$$

If a positive constant M is suitably chosen, we have, for any $\lambda_1>\lambda_0$,

$$\left|\, e^{-zt}F(\xi)\,\right| = \left|\, e^{(\lambda_1-z)\xi}\,\right|\cdot\left|\, e^{-\lambda_1\xi}F(\xi)\,\right| \leqq Me^{(\lambda_1-u)\xi}.$$

Therefore the integral (8) is absolutely convergent in the region $u>\lambda_0$ and uniformly convergent in the closed region $u\geqq\lambda>\lambda_0$. Hence the function $f(z)$ is analytic in the region $u>\lambda_0$.

Suppose now that the number $\lambda>\lambda_0$ is fixed. Since the integral (8) is uniformly convergent for

$$u\geqq\lambda, \tag{9}$$

it follows that to an arbitrary positive ϵ there corresponds a positive constant X independent of z in (9) such that

$$\left|\int_X^\infty e^{-zt}F(\xi)d\xi\right| < \frac{\epsilon}{2}.$$

Further, this constant X being fixed, the integral

$$\int_0^X e^{-zt}F(\xi)d\xi=\frac{e^{-zt}}{-z}F(\xi)\Big|_0^X+\frac{1}{z}\int_0^X e^{-zt}F'(\xi)d\xi$$

likewise is $<\epsilon/2$ in absolute value for z in (9) and $|z|$ sufficiently large. Thus the condition (A_1) is satisfied by $f(z)$. Denote now by α_1 an arbitrary positive number $<\alpha_0$. If (B_1) is satisfied, the formula (8) may be rewritten in either of the following forms:

$$(10_1)\quad f(z)=\int_0^{\alpha_1}e^{-zi\eta}F(i\eta)id\eta+\int_0^\infty e^{-z(\xi+i\alpha_1)}F(\xi+i\alpha_1)d\xi\equiv\varphi_1(z)+f_1(z)\ ;$$

$$(10_2)\quad f(z)=-\int_0^{\alpha_1}e^{zi\eta}F(-i\eta)id\eta+\int_0^\infty e^{-z(\xi-i\alpha_1)}F(\xi-i\alpha_1)d\xi\equiv\varphi_2(z)+f_2(z).$$

We shall proceed by associating the first form with the case $v\leqq 0$ and the second form with the case $v\geqq 0$. Accordingly, the occurence of the functions

$\varphi_k(z)$, $f_k(z)$ in a formula shall imply the former case when $k=1$, and the latter case when $k=2$.

Let us suppose for the moment that $F(0)=0$. Integration by pa ts gives

$$\varphi_1(z) = \int_0^{\alpha_1} e^{-zi\eta} F(i\eta) i d\eta = -\frac{e^{-zi\alpha_1}}{z} F(i\alpha_1) + \frac{i}{z}\int_0^{\alpha_1} e^{-zi\eta} F'(i\eta) d\eta = O\left(\frac{1}{z}\right),$$

because

$$|e^{-zi\eta}| = e^{v\eta} \leqq 1 \text{ for } v \leqq 0 \leqq \eta.$$

Integrating by parts once more we find easily for large $|z|$ and $|v|$:

$$(11_1) \qquad \varphi_1(z) = O\left(\frac{e^{-\alpha_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

The same is true of the function $\varphi_2(z)$:

$$(11_2) \qquad \varphi_2(z) = O\left(\frac{e^{-\alpha_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right).$$

It is obvious finally that $\varphi_1(z)$ and $\varphi_2(z)$ are entire transcendental functions of z.

The formulas $(10_{1,2})$ and $(11_{1,2})$ show that the functions $f_1(z)$ and $f_2(z)$ satisfy the condition (A_1). Moreover, denoting by λ_1 any number between λ_0 and λ we can write

$$f_1(z) = \int_0^\infty e^{-z(\xi+i\alpha_1)} F(\xi + i\alpha_1) d\xi = e^{-zi\alpha_1}\int_0^\infty e^{(\lambda_1 - z)\xi} e^{-\lambda_1\xi} F(\xi + i\alpha_1) d\xi,$$

which gives

$$(12) \qquad |f_1(z)| \leqq e^{-\alpha_1|v|}\int_0^\infty e^{(\lambda_1-\lambda)\xi} |e^{-\lambda_1\xi} F(\xi + i\alpha_1)| d\xi.$$

By virtue of (B_1)

$$|e^{-\lambda_1\xi} F(\xi + i\alpha_1)| = |e^{-\lambda_1(\xi+i\alpha_1)} F(\xi + i\alpha_1)| \to 0 \text{ as } \xi \to \infty.$$

Then (12) shows that

$$(13_1) \qquad f_1(z) = O(e^{-\alpha_1|v|})$$

for large values of $|v|$. In precisely the same way we find that

$$(13_2) \qquad f_2(z) = O(e^{-\alpha_1|v|}).$$

Suppose now that

$$F(0) = c \neq 0.$$

We have

$$f(z) = \frac{c}{z} + \int_0^\infty e^{-z\xi}\Big\{F(\xi) - F(0)\Big\}d\xi.$$

The integral being of the form discussed above, we see that

$$f(z) = \frac{c}{z} + \begin{cases} \bar{\varphi}_1(z) + \bar{f}_1(z) \text{ for } v \leqq 0, \\ \bar{\varphi}_2(z) + \bar{f}_2(z) \text{ for } v \geqq 0, \end{cases}$$

where the functions $\bar{\varphi}_1, \bar{\varphi}_2, \bar{f}_1, \bar{f}_2$ possess the same properties as the functions $\varphi_1, \varphi_2, f_1, f_2$.

Hence the function $f(z)$ satisfies the condition (A_4) and a fortiori the condition (A_2).

3. In this section we return to the point of view that the function $F(x)$ is unknown and $f(z)$ is given. We suppose now, however, that the function $f(z)$ satisfies the condition

(C) The function $f(z)$ may be represented in the form

$$f(z) = \frac{c}{z} + \begin{cases} \psi_1(z) + \theta_1(z) \text{ for } v \leqq 0, \\ \psi_2(z) + \theta_2(z) \text{ for } v \geqq 0, \end{cases}$$

where $\psi_1(z)$, $\psi_2(z)$ are entire transcendental functions which in the corresponding half-planes are of the order

$$\psi_k(z) = O\left(\frac{e^{-\alpha_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right) \qquad (k = 1,2) \tag{14}$$

for large $|z|$ and $|v|$, and where $\theta_1(z)$ and $\theta_2(z)$ satisfy the condition (A_1) and are of the order

$$\theta_k(z) = O(e^{-\alpha_1|v|}) \qquad (k = 1,2), \tag{15}$$

α_1 being an arbitrary positive number $< \alpha_0$.

Obviously $f(z)$ satisfies the conditions (A_1) and (A_2). Hence the solution of the equation (1) is given by the formula

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta)d\zeta.$$

We shall show now that, because of the special properties of the function $f(z)$, this solution can be extended to complex values of x and is analytic in x.

Suppose for the sake of simplicity that $c=0$ and denote by (D_1) the part of the contour (D) which is situated below the axis of reals and by

(D_2) the part which is above the axis of reals. The formulas (15), in which α_1 is an arbitrary positive number $< \alpha_0$, show that both integrals

$$F_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\zeta x} \theta_k(\zeta) d\zeta \qquad (k = 1,2) \tag{16}$$

converge absolutely for $|\eta| < \alpha_0$ and uniformly in any finite part of the strip $|\eta| \leqq \alpha$, α being an arbitrary positive number $< \alpha_0$. Thus $F_k(x)$ are analytic on the strip $|\eta| < \alpha_0$.

We shall prove that both functions $F_k(x)$ satisfy the condition (B_1). Let λ_1 be any number $> \lambda_0$; we can locate the contour (D) so that $\lambda_0 < \lambda < \lambda_1$, and then, denoting by M a suitable positive constant, we have

$$\left| e^{-\lambda_1 x} F_k(x) \right| \leqq M e^{(\lambda - \lambda_1)\xi} \int_0^\infty e^{v(\eta - \alpha_1)} dv \to 0 \text{ as } \xi \to \infty ,$$

the limit being approached uniformly in any region

$$|\eta| \leqq \alpha < \alpha_1 < \alpha_0 .$$

Thus, α_1 and therefore α being arbitrary numbers $< \alpha_0$, the property (B_1) is proved.

We turn now to the functions

$$\Phi_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\zeta x} \psi_k(\zeta) d\zeta \qquad (k = 1,2). \tag{17}$$

In this form the functions $\Phi_k(x)$ can not be extended to complex values of x. This becomes possible, however, if the contour of integration is suitably transformed. We shall show that

$$\Phi_1(x) = \frac{1}{2\pi i} \int_{-\infty}^{\lambda} e^{\zeta x} \psi_1(\zeta) d\zeta ; \qquad \Phi_2(x) = \frac{1}{2\pi i} \int_{\lambda}^{-\infty} e^{\zeta x} \psi_2(\zeta) d\zeta . \tag{18}$$

In order to prove this we observe that both functions ψ_1, ψ_2 are entire transcendental functions and that the integral

$$\int e^{\zeta x} \psi_k(\zeta) d\zeta$$

taken round the quarter of the circle $|\zeta - \lambda| = R$ which lies to the left of (D_k), approaches zero as $R \to \infty$, as follows from (14) using Jordan's lemma.*

* Whittaker and Watson, *Modern Analysis*, 1920, p. 115.

The formulas (18) combined with (14) show that the $\Phi_k(x)$ are analytic for $\xi>0$, and it is readily found that, for $\lambda_1>\lambda$,

$$| e^{-\lambda_1 x}\Phi_k(x) | < Me^{(\lambda-\lambda_1)\xi} \to 0 \text{ as } \xi \to \infty ,$$

the convergence being uniform on any strip $|\eta| <\alpha <\alpha_0$, $\xi>0$. The case when $c\neq 0$ does not involve any substantial change in the reasoning above. Thus the following theorem is proved:

THEOREM 2. *If the function $f(z)$ satisfies the condition* (C), *the solution $F(x)$ of the equation* (1) *satisfies the condition* (B) *which is identical with the condition* (B_1) *above except that the strip $\xi\geqq 0$, $|\eta| < \alpha_0$, closed at the left hand end, is replaced by the open strip*

$$\xi > 0, \quad |\eta| < \alpha_0.$$

We turn now to the converse theorem:

THEOREM 3. *If the function $F(x)$ satisfies the condition* (B_1), *the function $f(z)$ defined by*

$$f(z) = \int_0^\infty e^{-z\xi}F(\xi)d\xi$$

is a solution of the equation (2) *for $\xi>0$ and satisfies the condition* (C).*

This theorem establishes the reciprocity between $F(x)$ and $f(z)$, which was mentioned in the beginning of the paper.

It was proved in § 2 that the function

$$f(z) = \int_0^\infty e^{-z\xi}F(\xi)d\xi$$

satisfies the condition (C). It remains only to prove that $f(z)$ is a solution of the equation

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta .$$

We suppose again that

$$c = F(0) = 0$$

and consider first the case when x is real and positive. We have then

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x}f(\zeta)d\zeta = \frac{1}{2\pi i}\lim_{V\to\infty}\int_{\lambda-iV}^{\lambda+iV} e^{\zeta x}d\zeta\int_0^\infty e^{-\zeta\xi}F(\xi)d\xi .\dagger$$

* If x is complex, the right hand member of (2) must be transformed as was indicated above.

† See last foot note on p. 417.

Because of the uniform convergence of the interior integral we are justified in interchanging the order of integration, which gives

$$\frac{1}{2\pi i}\int_{(D)} e^{\zeta x} f(\zeta)\,d\zeta = \lim_{V\to\infty}\int_0^\infty F(\xi)e^{\lambda(x-\xi)}\int_{-V}^{V} e^{iv(x-\xi)}\,i\,dv$$

$$= \frac{1}{\pi}\lim_{V\to\infty}\int_0^\infty F(\xi)e^{\lambda(x-\xi)}\,\frac{\sin V(x-\xi)}{x-\xi}\,d\xi = F(x).$$

This last relation can be easily proved by using the Dirichlet formula. * Thus the equation (2) is proved for real positive values of x. The right hand member of this equation is an analytic function of x on the region $\xi>0$, as was proved above. The same is true by hypothesis of the left hand member. Hence the equation holds true on the whole region $\xi>0$.

The case $c\neq 0$ does not involve any substantial change in the applied reasoning; neither does the case in which finite sums of terms of the form

$$c_s z^{p_s}\ ;\quad a_s x^{-p_s-1} \qquad (\text{Re } p_s < 0)$$

are present in $f(z)$ and $F(x)$ respectively.

Under the restrictions made we can easily prove the formulas

$$F(x) = \frac{1}{2\pi i}\int_{(D)} e^{\zeta x}\,d\zeta\int_0^\infty e^{-\zeta\xi}F(\xi)\,d\xi\ ;$$

$$f(z) = \frac{1}{2\pi i}\int_0^\infty e^{-z\xi}\,d\xi\int_{(D)} e^{\zeta\xi}f(\zeta)\,d\zeta.$$

There is no difficulty either in proving that the solutions of the equations (1), (2), obtained above, are unique under the same restrictions.

* See Hamburger, loc. cit.

PETROGRAD

ON VOLTERRA'S INTEGRO-FUNCTIONAL EQUATION*

BY

J. D. TAMARKIN

Although the equation

$$u(x) = f(x) + s(x)u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u[\theta_2(\xi)]d\xi \tag{I}$$

has been discussed by many authors,† nevertheless the question can hardly be considered as completely solved. Hence the following considerations may present points of interest.

1. In the equation (I)

$$K(x, \xi), \quad f(x), \quad s(x), \quad \theta_1(x), \quad \theta_2(x)$$

are given functions which satisfy the following conditions:

(i) *The kernel $K(x, \xi)$ is determined and bounded in the region*

$$-X \leqq x \leqq X; \qquad -X \leqq \xi \leqq X,$$

where X is a given positive constant. The discontinuities of $K(x, \xi)$, if there are any, are regularly distributed.‡ The upper bound of $|K(x, \xi)|$ is denoted by k.

(ii) *The functions $\theta_1(x)$, $\theta_2(x)$, $s(x)$, $f(x)$ are continuous on the interval*

$$-X \leqq x \leqq X$$

and

$$|\theta_i(x)| \leqq |x| \ (i = 1, 2); \qquad |s(x)| \leqq \sigma; \quad |f(x)| \leqq f_0$$

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* Presented to the Society May 1, 1926; received by the editors in November, 1924.

† (a) Volterra, *Sopra alcuni questioni di inversione di integrali definiti*, Annali di Matematica, ser. 2, vol. 25 (1897); (b) E. Picard, *Sur une équation fonctionnelle*, Comptes Rendus, vol. 144 (1907), pp. 1009–1012; (c) T. Lalesco, *Sur l'équation de Volterra*, Journal de Mathématiques, ser. 6, vol. 4 (1908), pp. 156–160; (d) M. Picone, *Sopra un problema dei valori al contorno . . .*, Rendiconti del Circolo Matematico di Palermo, vol. 31 (1911), pp. 133–142; (e) A. Myller, *Randwertaufgaben bei partiellen Differential Gleichungen*, Mathematische Annalen, vol. 68 (1910), pp. 80–95; (f) C. Popovici, *Sur une équation fonctionnelle*, Comptes Rendus, vol. 158 (1914), pp. 1867–1869; (g) C. Popovici, *Nouvelles solutions de l'équation de Volterra* Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 341–344; (h) P. Nalli, *Sopra un'equazione funzionale*, Rendiconti della Reale Accademia dei Lincei, ser. 5, vol. 29 (2d sem. 1920), pp. 23–25, 84–86; vol. 30 (2d sem. 1921), pp. 85–90, 122–127; vol. 31 (1st sem. 1922), pp. 245–248; (i) P. Nalli, *Sopra un' equazione funzionale . . .*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 1–14.

‡ Hobson, *On the linear integral equation*, Proceedings of the London Mathematical Society, ser. 2, vol. 13 (1914), pp. 307–340.

where σ and f_0 are given positive constants.

The "solution" of the equation (I) is defined as a function which is integrable in the sense of Lebesgue and remains so if x is replaced by $\theta_i(x)$, and which satisfies the equation. The integrals are to be taken in the sense of Lebesgue.

Instead of the given equation (I) we shall consider the more general equation

$$\text{(II)} \qquad u(x) = f(x) + \lambda\left\{s(x)u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u[\theta_2(\xi)]d\xi\right\}$$

which reduces to (I) for $\lambda = 1$. This equation we shall try to satisfy by a power series in λ:

$$\text{(1)} \qquad u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x)$$

where $u_0(x), u_1(x), \cdots, u_n(x), \cdots$ are to be determined successively by the formulas

$$\text{(III)} \quad u_0(x) = f(x), \cdots, u_n(x) = s(x)u_{n-1}[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u_{n-1}[\theta_2(\xi)]d\xi.$$

The functions (III) are obviously all continuous. If we denote by $\bar{u}_n(x)$ the upper bound of $|u_n(\xi)|$ on the interval $-x \leqq \xi \leqq x$, it is readily found that

$$\bar{u}_0(x) \leqq f_0, \cdots, \bar{u}_n(x) \leqq \sigma\bar{u}_{n-1}(x) + 2k\int_0^x \bar{u}_{n-1}(\xi)d\xi.^*$$

Defining the functions

$$U_0(x), \quad U_1(x), \cdots, \quad U_n(x), \cdots$$

by the relations

$$U_0(x) = f_0, \cdots, U_n(x) = \sigma U_{n-1}(x) + 2k\int_0^x U_{n-1}(\xi)d\xi$$

the series

$$\text{(2)} \qquad U(x) = \sum_{n=0}^{\infty} \lambda^n U_n(x)$$

is clearly dominant for the series (1). On the other hand the series (2) is obtained if we formally expand the solution of the equation

$$U(x) = f_0 + \lambda\left\{\sigma U(x) + 2k\int_0^x U(\xi)d\xi\right\}$$

* For the sake of simplicity we consider the case $x \geqq 0$ only. The case $x \leqq 0$ can be treated in an entirely analogous way.

in powers of λ. This solution is uniquely determined and is given by the formula

$$U(x) = \frac{f_0}{1-\lambda\sigma}\, e^{\frac{2k\lambda x}{1-\lambda\sigma}},$$

which is expansible in the uniformly convergent power series in λ when $|\lambda| < 1/\sigma$. This series must coincide with the series (2) and is uniformly convergent for $\lambda = 1$, provided $\sigma < 1$. In this case the series (1) is also uniformly convergent for $\lambda = 1$ and yields the solution of (I). *Thus the existence of a continuous solution of* (I) *is proved, if in addition to the conditions* (i) *and* (ii) *we suppose that*

(iii) $$\sigma < 1.$$

It is easy to show now that *the solution of* (I), as defined above, *is unique, under the condition that it be bounded,* so that any bounded solution of (I) coincides with (1) (for $\lambda = 1$).

In order to prove this, it is sufficient to show that any bounded solution of the homogeneous equation

(3) $$v(x) = s(x)v[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)v[\theta_2(\xi)]d\xi$$

is identically zero. Let us denote by $\bar{v}(x)$ the upper bound of $|v(\xi)|$ on the interval $-x \leqq \xi \leqq x$. Then

$$\bar{v}(x) \leqq \sigma\bar{v}(x) + 2k\int_0^x \bar{v}(\xi)d\xi.$$

If v_0 is the upper bound of $\bar{v}(x)$ on the whole interval $(-X, X)$, we obtain successively

$$\bar{v}(x) \leqq v_0\frac{2kx}{1-\sigma}; \quad \bar{v}(x) \leqq v_0\left(\frac{2kx}{1-\sigma}\right)^2\frac{1}{2!}; \cdots; \bar{v}(x) \leqq v_0\left(\frac{2kx}{1-\sigma}\right)^n\frac{1}{n!}; \cdots$$

whence

$$\bar{v}(x) \leqq \frac{v_0}{n!}\left(\frac{2kX}{1-\sigma}\right)^n$$

for all n, however large. This is possible only if

$$\bar{v}(x) \equiv 0, \quad \text{that is,} \quad v(x) \equiv 0.$$

The same results hold true if we drop the condition of the continuity of the functions $s(x)$, $\theta_i(x)$, $f(x)$, and suppose only that

(iv) *The functions* $s(x)$, $\theta_i(x)$, $f(x)$ *are measurable and bounded and all the terms of the sequence* (III) *are measurable and remain measurable if* x *is replaced by* $\theta_i(x)$.

Under the conditions (i)–(iv) *the bounded solution of* (I) *is unique and is given by the formulas above.*

2. Consider now the case $\sigma \geqq 1$, under the hypotheses

(v) $$|\theta_1(x)| \leqq \alpha |x| ; \qquad \alpha\sigma < 1.^*$$

As we shall see below, the equation (I), if $\sigma \geqq 1$, may possess infinitely many continuous solutions. Therefore in this section we confine our discussion to the continuous solutions of (I) for which the ratio $(u(x)-u(0))/x$ remains bounded as $x \to 0$.

If $s(0) \neq 1$, the equation (I) determines uniquely the initial value $u(0)$:

$$u(0) = \frac{f(0)}{1 - s(0)}.$$

If $s(0) = 1$, it is necessary for the existence of a solution that $f(0) = 0$, and if this condition is satisfied, then $u(0)$ may be chosen arbitrarily. We introduce now the further restriction:

(vi) *The ratios*

$$\frac{f(x) - f(0)}{x}, \qquad \frac{s(x) - s(0)}{x}$$

remain bounded as $x \to 0$, *and*

$$f(0) = 0 \quad \text{if} \quad s(0) = 1.$$

Introducing in (I) a new dependent variable $z(x)$ given by

$$z(x) = \frac{u(x) - u(0)}{x} ; \quad u(x) = u(0) + xz(x),$$

we obtain

(4) $$z(x) = \varphi(x) + r(x)[\theta_1(x)] + \int_{-x}^{x} L(x, \xi) z[\theta_2(\xi)]\, d\xi,$$

where

$$\varphi(x) = \begin{cases} \dfrac{u(0)}{x}\displaystyle\int_{-x}^{x} K(x, \xi)\, d\xi + \dfrac{f(x) - f(0) + s(x)f(0) - s(0)f(x)}{x[1 - s(0)]}, & \text{if } s(0) \neq 1 ; \\ \dfrac{u(0)}{x}\displaystyle\int_{-x}^{x} K(x, \xi)\, d\xi + \dfrac{f(x)}{x} + \dfrac{s(x) - 1}{x} u(0), & \text{if } s(0) = 1 ; \end{cases}$$

$$r(x) = s(x)\frac{\theta_1(x)}{x} ; \qquad L(x, \xi) = K(x, \xi)\frac{\theta_2(\xi)}{x}.$$

* This implies, of course, $\alpha < 1$.

The functions $\varphi(x)$, $r(x)$, $L(x, \xi)$ replace the functions $f(x)$, $s(x)$, $K(x, \xi)$ of §1. By virtue of (v) and (vi) they are bounded and

$$|r(x)| \leqq \alpha\sigma < 1.$$

It is obvious that the conditions (i)–(iv) are satisfied for the equation (4) and therefore the equation (4) admits of a unique bounded solution, $z(x)$, which is given by the method above.

Thus, under the conditions (i)–(iii), (v), (vi), *the equation* (I) *possesses a continuous solution for which the ratio* $(u(x)-u(0))/x$ *remains bounded as* $x\to 0$. *If* $s(0)\neq 1$, *this solution is unique. If* $s(0)=1$, *the initial value* $u(0)$ *can be chosen arbitrarily, and when* $u(0)$ *is prescribed the solution is uniquely determined.*

3. A series of examples in which the theorem of uniqueness for bounded solutions fails, was given by C. Popovici.* The example which follows is simpler and apparently more general.† We consider a particular case of the homogeneous equation (I), namely

$$u(x) = \sigma u(\alpha x); \qquad 0 < \alpha < 1; \qquad x > 0. \tag{5}$$

The most general solution of (5) can be found explicitly. Setting

$$y = \log x; \quad \mu = \log \sigma; \quad \nu = \log \alpha; \quad v(\log x) = \log u(x),$$

we get

$$v(y) = \mu + v(y + \nu),$$

which gives

$$v(y) = -\frac{\mu}{\nu} y + \varpi(y),$$

where $\varpi(y)$ is an arbitrary periodic function of y, of period ν. Hence the general solution of (5) is

$$u(x) = x^{-\mu/\nu} e^{\tilde{\omega}(\log x)} \tag{6}$$

Let $\omega(x)$ be a continuous function which is not constant. Then $e^{\tilde{\omega}(\log x)}$ has positive lower and upper bounds on the interval $(0, \infty)$ and tends to no limit as $x\to 0$. The corresponding solution $u(x)$, which is given by (6), is not bounded as $x\to 0$, because $-\mu/\nu < 0$. This was to be expected, for it was proved above that the bounded solution of (5) is identically zero.

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* Loc. cit.

† This example was indicated by A. Friedmann.

Suppose now that $\sigma \geqq 1$, but $\alpha\sigma < 1$. Since $-\mu/\nu > 0$ the formula (6) gives us infinitely many solutions which are bounded and even continuous if we set $u(0) = u(0+) = 0$.

For all these solutions, however, the ratio $(u(x) - u(0))/x$ is not bounded. Suppose finally $\alpha\sigma \geqq 1$. In this case all the solutions determined by (6) are bounded and continuous (if we set $u(0) = 0$) and for all these solutions the ratio $(u(x) - u(0))/x$ remains bounded. The particular example of the equation

$$u(x) = \sigma u\left(\frac{x}{\sigma}\right)$$

which admits of infinitely many linear solutions

$$u(x) = Cx, \qquad C \text{ constant},$$

shows that there is no question about the theorem of uniqueness in this case.

Analogous results may be obtained for the more general equation

$$u(x) = \sigma u(\alpha x) + \int_0^x K(x, \xi) u(\xi) d\xi.$$

PETROGRAD

ON THE DISCRIMINANT OF TERNARY FORMS AND A CERTAIN CLASS OF SURFACES*

BY
ARNOLD EMCH

I. Introduction

In a paper *On the Weddle surface and some analogous loci*† I have studied the class of surfaces and curves defined as the loci of vertices of general cones passing through certain definite numbers of generic fixed points in space.

This investigation may be extended to such systems of cones with nodal generatrices. For this purpose it is obviously necessary to make use of the properties of the discriminant of a ternary form, to which the intersection of a cone with a generic plane reduces.

II. The discriminant of a ternary form

Let

$$C_n = \sum a_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma = 0, \quad \alpha+\beta+\gamma = n, \tag{1}$$

be an irreducible plane n-ic; then the discriminant Δ of C_n is a homogeneous polynomial of degree $3(n-1)^2$ in the coefficients. If

$$a^{p_1}_{\alpha_1\beta_1\gamma_1}\ a^{p_2}_{\alpha_2\beta_2\gamma_2}\ a^{p_3}_{\alpha_3\beta_3\gamma_3}\cdots a^{p_k}_{\alpha_k\beta_k\gamma_k}, \tag{2}$$

$p_1+p_2+p_3+\cdots+p_k=n$, denotes the general term of Δ, the indices and exponents of the a's have the property that each term of Δ has the same weight $n(n-1)^2$ with respect to each of the three indices,‡ i. e.

$$\begin{aligned}\alpha_1p_1+\alpha_2p_2+\cdots+\alpha_kp_k &= \beta_1p_1+\beta_2p_2+\cdots+\beta_kp_k \\ &= \gamma_1p_1+\gamma_2p_2+\cdots+\gamma_kp_k = n(n-1)^2.\end{aligned} \tag{3}$$

III. The n-ic through the projections from a generic point in space of $N=n(n+3)/2$ points upon a plane

Following a procedure similar to that of the paper mentioned above, let $A_i(a_1^i a_2^i a_3^i a_4^i)$, $i=1, 2, \cdots, N$, be the fixed points and $P(x_1x_2x_3x_4)$ the center of projection, so that the projection of A_i upon $x_4=0$ is

* Presented to the Society, May 1, 1926; received by the editors in December, 1925.

† These Transactions, vol. 27 (1925), pp. 270–278.

‡ B. Bauer, *Vorlesungen über Algebra*, 1910, pp. 88–97.

$A_i'(a_k^i x_4 - a_4^i x_k)$, $k=1, 2, 3$, or using cartesian coördinates for the points in space, $A_i'(x-a_i, y-b_i, z-c_i)$. The plane n-ic through the points A_i' is in determinant form

$$(4)\quad \begin{vmatrix} X^n & X^{n-1}Y & \cdots & X^\alpha Y^\beta Z^\gamma & \cdots & Z^n \\ (x-a_1)^n & (x-a_1)^{n-1}(y-b_1) & \cdots & (x-a_1)^\alpha (y-b_1)^\beta (z-c_1)^\gamma & \cdots & (z-c_1)^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (x-a_N)^n & (x-a_N)^{n-1}(y-b_N) & \cdots & (x-a_N)^\alpha (y-b_N)^\beta (z-c_N)^\gamma & \cdots & (z-c_N)^n \end{vmatrix} = 0.$$

Thus, denoting the reduced expression by

$$(5)\qquad \sum a_{\alpha\beta\gamma} X^\alpha Y^\beta Z^\gamma = 0,$$

and following a procedure similar to that of my previous paper, from (4) it is seen that the coefficient $a_{\alpha\beta\gamma}$ is of degree $\frac{1}{6}n(n+1)(n+2)-\alpha$ in x, $\frac{1}{6}n(n+1)(n+2)-\beta$ in y, and $\frac{1}{6}n(n+1)(n+2)-\gamma$ in z.

IV. Curves with singularities

We now construct the discriminant of (5) which has the form expressed by (2). According to the result just stated in the foregoing sections, its degree in x will be that of $a_{\alpha_1\beta_1\gamma_1}^{p_1} \cdot a_{\alpha_2\beta_2\gamma_2}^{p_2} \cdots$ in x, or

$$\sum [\tfrac{1}{6}n(n+1)(n+2) - \alpha_i]p_i = \tfrac{1}{6}n(n+1)(n+2)3(n-1)^2 - n(n-1)^2.$$

As this may be written in the form

$$\frac{1}{3}3(n-1)^2 n\frac{(n+3)}{2}$$

it follows that *the degree in x (and also in y and z) is equal to one third times the degree of the discriminant of a ternary n-ic, times the order of the n-ic, times the number of points necessary to determine a plane n-ic.*

The expression Δ in its reduced form is a polynomial in x, y, z of this same degree, since $(y=0, z=0)$, which is a generic line with respect to $\Delta=0$, cuts the surface $\Delta=0$ in as many points as there are roots in the resulting equation $\Delta(x, 0, 0)=0$. As the coefficients of Δ contain each triple $x-a_i, y-b_i, z-c_i$ homogeneously of degree n, the fixed points A_i are multiple points of order $3n(n-1)^2$ of $\Delta=0$. Moreover a point on the join of two A_i's is the vertex of $3(n-1)^2$ n-ic cones with singular generatrices. Hence the

Theorem. *The locus of vertices of n-ic cones with nodal generatrices through $\frac{1}{2}n(n+3)$ generic points in space is a surface of order $\frac{1}{2}(n-1)^2n^2(n+3)$, which has each of these points as a multiple point of order $3n(n-1)^2$, and each join of two of these points as multiple line of order $3(n-1)^2$.*

V. Examples

In case of five points, $n=2$, the order of $\Delta=0$ is $w=10$. This can easily be verified geometrically. The five points in groups of three can be joined by 10 planes. Each plane with any plane through the join of the two remaining points forms a degenerate cone through the five points. Thus the first of these two planes may be considered as the locus of vertices of ∞^1 degenerate cones. The same is true for each of the remaining 9 planes. Through the join of two of the five points pass three of these planes, so that this join is a triple line of the surface. Through each of the five points there are six of these planes, so that the point is six-fold for the surface. These numbers are in agreement with the numbers in the general case.

Cubic cones. For $n=9$, we may restate the

Theorem. *The locus of vertices of cubic cones with singular generatrices through* 9 *points is a surface of order* 108, *with* 36*-fold points at these points and with each join of two as a* 12*-fold line.*

A plane p through three of the 9 points cuts the surface in a plane curve of order 108. But each of the three sides of the triangle determined by the three points must be counted 12 times in the intersection, so that the curve of intersection proper is of order $108-48=60$ which has 12-fold points at each of the $(6\cdot 5)/2=15$ points of intersection of the joins of the six points outside of p with p. The Weddle surface determined by the six points cuts p in a quartic which passes singly through each of the 15 points. This is a component part of the curve of order 60 because each point of this quartic is the vertex of a quadric cone which together with p forms a degenerate cubic cone. The residual curve of vertices in p of non-degenerate singular cubic cones is therefore a curve of order 56 with 11-fold points at the 15 points.

In the previous paper the existence of a curve R_{12} is established from whose points the 9 points are projected into 9 base points of a pencil of cubics. There are 12 singular cubics in every pencil. Consequently every point of R_{12} is the vertex of 12 cubic cones with singular generatrices. From this it follows that *the surface of order* 108 *has* R_{12} *as a* 12*-fold singular line.*

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ON THE CONVERGENCE OF CERTAIN METHODS OF CLOSEST APPROXIMATION*

BY

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1. Introduction. Let $f(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$. Let $p_1(x), p_2(x), \cdots, p_n(x)$ be n functions of x, continuous and linearly independent in this interval. Then there exists a choice of the coefficients c_k in

$$\phi(x) = c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x)$$

such that, for m any number greater than or equal to one, the integral

$$\int_a^b |f(x) - \phi(x)|^m dx$$

has its minimum value.† If the coefficients are given these particular values, then $\phi(x)$ is called an *approximating function for $f(x)$ corresponding to the exponent m.*

The convergence of the approximating function to $f(x)$ as n becomes infinite and m is held fast, has been investigated‡ in the cases where $\phi(x)$ is a polynomial or a trigonometric sum. It is the purpose of this paper to extend this work to a more general class of approximating functions. In § 2, a finite Sturm-Liouville sum of order n is taken as the approximating function; in § 3 the Sturm-Liouville sum is replaced by a linear combination of characteristic solutions of a certain third order differential equation with a particular set of boundary conditions. In § 4, two extensions of Bernstein's theorem on the derivative of a finite trigonometric sum are given.

2. The convergence of the Sturm-Liouville approximating function. Let $f(x)$ be a continuous function of x in $0 \leqq x \leqq \pi$. Let $v_0, v_1, \cdots, v_n$ be the first $(n+1)$ characteristic solutions of the Sturm-Liouville system

$$v'' + (\lambda + l(x))v = 0,$$

$$v'(0) - hv(0) = 0, \quad v'(\pi) + Hv(\pi) = 0,$$

* Presented to the Society, April 19, 1924; received by the editors in December, 1925.

† D. Jackson, *On functions of closest approximation*, these Transactions, vol. 22 (1921), pp. 117–128; also *Note on a class of polynomials of approximation*, these Transactions, vol. 22 (1921), pp. 320–326. The first of these papers will be referred to below as A and the second as B.

‡ Jackson, *On the convergence of certain trigonometric and polynomial approximations* (referred to below as C), these Transactions, vol. 22 (1921), pp. 158–166.

where $l(x)$ has a continuous second derivative for $0 \leqq x \leqq \pi$. The functions $v_k(x)$ are mutually orthogonal with respect to the interval $(0, \pi)$ and hence are linearly independent in this interval. Since they are linearly independent and continuous, it follows that corresponding to every value of $m \geqq 1$, there exists* an approximating function for $f(x)$ of the form

$$S_n(x) = c_0 v_0(x) + c_1 v_1(x) + \cdots + c_n v_n(x).$$

If $m = 2$, the coefficients are the Sturm-Liouville coefficients.†

To determine sufficient conditions on $f(x)$ in order that for a fixed m $(m \geqq 1)$, the approximating function $S_n(x)$ will converge uniformly to the value of $f(x)$ as n becomes infinite, all that is required is to bring together the necessary materials already in the literature on the subject. The general method of proof is the same as that used in proving the convergence of the trigonometric approximating function.‡ There are just two differences worthy of mention:

(i) In the trigonometric case, the proof depends upon Bernstein's theorem for the derivative of a finite trigonometric sum; in the Sturm-Liouville case, the proof depends upon an extension of Bernstein's theorem to Sturm-Liouville sums.§ The presence of a constant multiplier p in the latter theorem, which does not appear in the theorem for the trigonometric case, having the value unity there, does not require any change in the method of proof.

(ii) Instead of the theorems on the degree of approximation possible by means of trigonometric sums, the corresponding theorems on approximation by Sturm-Liouville sums‖ find application in the proof for the present case.

The following theorems can be stated for the convergence of the Sturm-Liouville approximating function $S_n(x)$, the demonstration being supplied

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* See papers A and B. The uniqueness of the approximating function is proved only for $m > 1$, but if there is more than one approximating function when $m = 1$, any one of these can be selected for each value of n, and the convergence of the sequence thus obtained can be studied in the same way as when $m > 1$.

† For the proof of this statement cf. the proof of the corresponding fact in the trigonometric case, M. Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, vol. 7 (1906), pp. 81–152; and J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, Journal für Mathematik, vol. 94 (1883), pp. 41–73.

‡ See paper C, pp. 159–161.

§ E. Carlson, *Extension of Bernstein's theorem to Sturm-Liouville sums*, these Transactions, vol. 26 (1924), pp. 230–240.

‖ Jackson, *On the degree of convergence of Sturm-Liouville series*, these Transactions, vol. 15 (1914), pp. 439–466.

in substance by the papers cited. (The results will be stated in a form resulting from the direct use of the theorems of the paper last mentioned, though the hypotheses could be lightened by means of suitable extensions of those theorems.)

The sum $S_n(x)$ converges uniformly to the value of $f(x)$ in $0 \leqq x \leqq \pi$:

For $m=1$, if $f(x)$ has a first derivative satisfying a Lipschitz condition throughout $0 \leqq x \leqq \pi$, and if $f(x)$ and $f'(x)$ vanish at 0 and at π;

For an arbitrary $m>1$, if $f(0)=0$ and if $f(x)$ has a modulus of continuity $\omega(\delta)$ such that

$$\lim_{\delta=0} \omega(\delta)(\log \delta)/\delta^{(1/m)} = 0.$$

The last condition is satisfied for any value of $m>1$ if $f(x)$ satisfies a Lipschitz condition.*

The above problem can be generalized by taking as the quantity to be reduced to a minimum the integral†

$$\int_0^\pi \psi(x) \mid f(x) - S_n(x) \mid^m dx,$$

where $\psi(x)$ is a positive continuous function of x in $0 \leqq x \leqq \pi$, or more generally, a bounded measurable function with a positive lower bound. The same conditions for convergence hold as in the case just discussed, and the proof needs only a few slight modifications.

3. **The convergence of a third order approximating function.** Let the characteristic solutions of the differential system

$$u'''(x) + \rho^3 u(x) = 0,$$
$$u''(0) = 0,\ u''(\pi) = 0,\ u'(0) + u'(\pi) = 0$$

be denoted by

$$u_0,\ u_{kj}(x) \qquad (k = 1,2,\cdots;\ j = 1,2).$$

For the sake of conciseness in presenting the main conclusions, certain facts with regard to this system and its characteristic solutions will be assumed here, to be discussed in more detail in the following section.

The functions $u_0, u_{kj}(x)$ are linearly independent and continuous in $0 \leqq x \leqq \pi$. Let

$$u_{k1}(x) = U_k + iV_k,$$

* For $m=2$, this becomes a theorem on the convergence of Sturm-Liouville series; cf. Kneser, Mathematische Annalen, vol. 58 (1904), pp. 81–147.

† See D. Jackson, *Note on the convergence of weighted trigonometric series*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 259–263.

where U_k and V_k are real. Then

$$u_{k2}(x) = U_k - iV_k$$

(cf. VII in § 4). The functions u_0, U_k, V_k are linearly independent and continuous in $(0, \pi)$. Because of these properties, there exists* for every real function $f(x)$, continuous in $(0, \pi)$, a linear combination $S_n(x)$ of the functions $u_0, U_1, \cdots, U_n, V_1, \cdots, V_n$, which reduces to a minimum the integral of the mth power $(m>1)$ of the absolute value of the difference $f(x)-S_n(x)$ (or, more generally, the integral of $\psi(x)$ times the mth power). This is with the understanding at first that all the coefficients in the linear combinations considered are real, but it is clear that no smaller value could be obtained for the integral by the admission of complex coefficients. So $S_n(x)$, which is at the same time a linear combination of u_0, u_{k1}, u_{k2} (with complex coefficients, in general), gives a better approximation to $f(x)$, as measured by the value of the integral, than any other linear combination of these functions with real or complex coefficients, since every such combination is a linear combination of u_0 and the U's and V's. The situation is similar when $m=1$, except that it is not clear that the approximating function is unique. Under suitable hypotheses on the function $f(x)$, $S_n(x)$ converges uniformly to the value of $f(x)$.

To determine sufficient conditions for the convergence of $S_n(x)$, we make use of the theorems on the degree of convergence of Birkhoff's series due to Milne,† and the extensions of Bernstein's theorem proved in the following section of this paper. The method used in the convergence proofs is the same as that used in the trigonometric case.‡

Applying the first theorem of § 4, together with Theorem 1 of Milne's paper, we can prove the following:

The approximating function $S_n(x)$ converges uniformly to $f(x)$ in $(0, \pi)$:

For $m=1$, if $f(x)$ has a continuous third derivative of limited variation in $(0, \pi)$ and if $f''(x)$, $f'(x)$, and $f(x)$ vanish at 0 and at π;

For $1<m\leqq 2$, if $f(x)$ has a continuous second derivative of limited variation in $(0, \pi)$ and if $f'(x)$ and $f(x)$ vanish at 0 and at π;

For $m>2$, if $f(x)$ has a continuous first derivative of limited variation in $(0, \pi)$, and vanishes at 0 and at π.

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* See papers A and B.

† W. E. Milne, *On the degree of convergence of Birkhoff's series*, these Transactions, vol. 19 (1918), pp. 143–156.

‡ Jackson, paper C.

For convergence in the interior of the interval $(0, \pi)$, less restrictive hypotheses on $f(x)$ are sufficient. Such conditions may be derived from the second theorem of § 4 together with the preceding work and the theorem of Milne's paper already referred to, the method being analogous to that of § 6 of the paper C. They relate to the uniform convergence of $S_n(x)$ to the value of $f(x)$ throughout an arbitrary interval (α, β) such that $0<\alpha<\beta<\pi$.

For $m=1$ (and any larger value of m) it is sufficient that $f(x)$ have a continuous second derivative of limited variation in $(0, \pi)$, and that $f'(x)$ and $f(x)$ vanish at 0 and at π;

For $m>\sqrt{2}$, it is sufficient that $f(x)$ have a continuous first derivative of limited variation in $(0, \pi)$ and vanish at 0 and at π.

These theorems, like those of the preceding section, are valid for the general case in which $S_n(x)$ is determined so as to minimize the integral

$$\int_0^\pi \psi(x) \left| f(x) - S_n(x) \right|^m dx$$

with a weight function $\psi(x)$ which is bounded and measurable and has a positive lower bound.

4. **Extension of Bernstein's theorem to a third order case.** The purpose of this section is to supply the extensions of Bernstein's theorem which are prerequisite to the work of the preceding section.

Consider the differential equation

$$u'''(x) + \rho^3 u(x) = 0 \tag{1}$$

in the interval $0 \leqq x \leqq \pi$, with the regular* boundary conditions

$$u''(0) = 0, \quad u''(\pi) = 0, \quad u'(0) + u'(\pi) = 0. \tag{2}$$

Let us denote the characteristic numbers of this system by

$$\rho_0, \quad \rho_{kj} \qquad (j = 1,2\,;\ k = 1,2, \cdots),$$

and the characteristic solutions by

$$u_0(x), \quad u_{kj}(x) \qquad (j = 1,2\,;\ k = 1,2, \cdots).$$

Properties of the characteristic numbers and solutions can be obtained by applying Birkhoff's† work, or, for the specific case in hand, can be derived

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* For definition, see Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395.

† See preceding footnote.

by processes of direct calculation materially simpler than those required for the treatment of the general problem.

The following facts are of importance in the subsequent work:

I. The characteristic numbers are all simple.

II. If $u(x)$ is a characteristic solution corresponding to $\rho=\rho'$, then $u(\pi-x)$ is a characteristic solution corresponding to $\rho=-\rho'$. Hence

$$\rho_{k2} = -\rho_{k1}$$

and

$$u_{k2}(x) = u_{k1}(\pi - x).$$

III. Zero is a characteristic number, and the corresponding characteristic solution is a constant, i. e., $\rho_0=0$ and $u_0(x)=$a constant, and as each characteristic function contains an arbitrary constant factor, $u_0(x)$ can be taken equal to 1.

IV. The system adjoint to (1) and (2) is

$$-v'''(x) + \rho^3 v(x) = 0,$$
$$v''(\pi) = 0, \quad v''(0) = 0, \quad v'(\pi) + v'(0) = 0,$$

and its characteristic solutions are

$$v_0(x) = u_0(x), \quad v_{k1}(x) = u_{k2}(x), \quad v_{k2}(x) = u_{k1}(x).$$

V. The following integral relations hold:

$$\int_0^\pi u_{kj}(x)v_{lm}(x)dx = 0, \qquad |k-l| + |j-m| \neq 0;$$

$$\int_0^\pi u_{kj}(x)v_{kj}(x)dx \neq 0;$$

$$\int_0^\pi u_{kj}(x)dx = 0, \qquad j = 1,2; \quad k \geqq 1;$$

$$\int_0^\pi u_0(x)dx \neq 0.$$

VI. The characteristic numbers are pure imaginaries.

VII. Because of I, II, and VI, and the fact that the coefficients in the differential system are real, ρ_{k2} is conjugate to ρ_{k1}, and $u_{k2}(x)$ is the conjugate of $u_{k1}(x)$.

VIII. Asymptotic forms for the characteristic numbers are

$$\rho_{k1} = -2i/3 + 2(k+k_0)i + \epsilon_k, \tag{3}$$
$$\rho_{k2} = 2i/3 - 2(k+k_0)i - \epsilon_k,$$

where k_0 is some integer and $\lim_{k\to\infty}\epsilon_k=0$, or more precisely*

$$|\epsilon_k| < \frac{b_1}{k}$$

for all values of k.

IX. With suitable choice of the arbitrary constant factors involved, the characteristic solutions are uniformly bounded functions of x in $0\leqq x\leqq\pi$ for all values of k, which can be expressed in the form

$$\begin{aligned} u_{k1}(x) &= e^{-\rho_{k1}x} + c_1e^{-\rho_{k1}\omega^2 x} + c_2e^{-\rho_{k1}\omega(x-\pi)} = v_{k2}(x), \\ v_{k1}(x) &= e^{\rho_{k1}x} + c_3e^{\rho_{k1}\omega x} + c_4e^{\rho_{k1}\omega^2(x-\pi)} = u_{k2}(x), \end{aligned} \tag{4}$$

where the c's depend upon k but are bounded for all values of k; $\omega=e^{2\pi i/3}$; c_3 is the conjugate of c_1, and c_4 is the conjugate of c_2.

X. If x is restricted to an interval interior to $(0, \pi)$, then the following asymptotic forms hold:

$$\begin{aligned} u_{k1}(x) &= v_{k2}(x) = e^{(2/3-2k-2k_0)ix} + E_{k1}/k, \\ u_{k2}(x) &= v_{k1}(x) = e^{(-2/3+2k+2k_0)ix} + E_{k2}/k, \end{aligned} \tag{5}$$

where E_{k1} and E_{k2} denote functions of k and x, uniformly bounded for all values of k. Each function E has the further property that its derivative with regard to x is of the form $k\bar{E}$, where $\bar{E}$ is a uniformly bounded function of k and x.

Let

$$\begin{aligned} S_n(x) = a_0u_0(x) &+ a_{11}u_{11}(x) + a_{21}u_{21}(x) + \cdots + a_{n1}u_{n1}(x) \\ &+ a_{12}u_{12}(x) + a_{22}u_{22}(x) + \cdots + a_{n2}u_{n2}(x), \end{aligned} \tag{6}$$

where the a's are arbitrary constants, real or complex. Because of the relations V,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi S_n(t)dt, \\ a_{kj} &= \frac{1}{D_{kj}}\int_0^\pi S_n(t)v_{kj}(t)dt, \qquad D_{kj} = \int_0^\pi u_{kj}(t)v_{kj}(t)dt. \end{aligned} \tag{7}$$

We may proceed to prove

THEOREM I. *If $|S_n(x)|\leqq M$ for $0\leqq x\leqq\pi$, then throughout this interval, $|S_n'(x)|\leqq n^2pM$, where p is independent of n and of the coefficients in $S_n(x)$.*

The proof† depends upon the two relations

* We shall use the letter b with subscripts to denote constants independent of k.

† Cf. M. Fekete, *Über einen Satz des Herrn Serge Bernstein*, Journal für die reine und angewandte Mathematik, vol. 146 (1916), pp. 88–94.

$$|u'_{kj}(x)| \leqq kb_2, \tag{8}$$

$$|a_{kj}| \leqq b_3 M, \tag{9}$$

for $j=1, 2$, and for all values of k; the factor b_3 is not merely independent of k, but is the same for all sums $S_n(x)$, regardless of the value of n. From (4) it follows that

$$|u'_{k1}(x)| \leqq |\rho_{k1}||e^{-\rho_{k1}x} + \omega^2 c_1 e^{-\rho_{k1}\omega^2 x} + \omega c_2 e^{-\rho_{k1}\omega(x-\pi)}|.$$

The second factor on the right in this inequality is uniformly bounded for all values of k, and from (3) it is seen that $|\rho_{k1}| \leqq b_4 k$ for all values of k; hence relation (8) holds for $j=1$. The proof for (8) when $j=2$ is exactly analogous.

To prove (9) we note from (4) that $u_{kj}(t)v_{kj}(t)$ may be expressed as the sum of nine terms, of which the first is 1, while each of the others contains a product of exponentials in which the real parts of the exponents are negative throughout the interior of the interval $(0, \pi)$, vanishing for $t=0$ in some cases and for $t=\pi$ in others. A typical term is $c_3 e^{-\rho_{k1}t} e^{\rho_{k1}\omega t}$. The absolute value of $e^{-\rho_{k1}t}$ does not exceed 1. In the other exponential, the real part of $\rho_{k1}\omega$ is negative and greater numerically than a constant multiple of k, so that $|e^{\rho_{k1}\omega t}| \leqq e^{-Ckt}$, where C is positive and independent of k. So

$$\left| \int_0^\pi e^{-\rho_{k1}t} e^{\rho_{k1}\omega t} dt \right| \leqq \int_0^\pi e^{-Ckt} dt = \frac{1}{Ck}(1 - e^{-Ck\pi}),$$

which is of the order $1/k$. Each of the remaining terms may be treated similarly, to show that

$$D_{kj} = \pi + g_1(k), \qquad |g_1(k)| < \frac{b_5}{k}. \tag{10}$$

Hence $|1/D_{kj}| < b_6$, and

$$|a_{kj}| \leqq b_6 \left| \int_0^\pi S_n(t) v_{kj}(t) dt \right| \leqq b_6 b_7 M = b_3 M.$$

By (8) and (9) and the definition of $S_n(x)$

$$\begin{aligned} |S'_n(x)| &\leqq \sum_{k=1}^n |a_{k1}||u'_{k1}(x)| + \sum_{k=1}^n |a_{k2}||u'_{k2}(x)| \\ &\leqq \sum_{k=1}^n b_3 M b_2 k + \sum_{k=1}^n b_3 M b_2 k \\ &\leqq n^2 p M \end{aligned}$$

as the theorem asserts.

Let $f(x)$ be a bounded measurable function in $(0, \pi)$, and let $S_n(x)$ be the linear combination of $u_0(x)$, $u_{11}(x), \cdots, u_{n1}(x)$, $u_{12}(x), \cdots, u_{n2}(x)$, having the coefficients

$$(11) \qquad a_0 = \frac{1}{\pi}\int_0^\pi f(t)dt, \qquad a_{kj} = \frac{1}{D_{kj}}\int_0^\pi f(t)v_{kj}(t)dt.$$

Let $\sigma_n(x)$ be the arithmetical mean of $S_1(x), \cdots, S_n(x)$:

$$\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}.$$

We shall lead up to a second theorem through the following

LEMMA. *If $|f(x)| \leqq M$ for $0 \leqq x \leqq \pi$, then for $\epsilon \leqq x \leqq \pi - \epsilon$, $|\sigma_n'(x)|$ can not exceed nq_1M, where q_1 is a constant independent of n and $f(x)$, but does depend upon the choice of ϵ.*

In the proof, we shall assume $a_0 = 0$. There is no loss of generality in doing so, for if the lemma holds when $a_0 = 0$, it holds also when $a_0 \neq 0$.

We need certain asymptotic forms for a_{k1} and a_{k2}, which can be obtained by substituting the values for v_{k1} and v_{k2} given in (4) in the formulas (11) defining a_{k1} and a_{k2}. In this way we get

$$(12) \qquad a_{k1} = \frac{1}{\pi}\int_0^\pi f(t)e^{(-2/3+2k+2k_0)it}dt + g_2, \quad |g_2| < \frac{Mb_8}{k},$$

for the terms $c_3e^{\rho_{k1}\omega x}$ and $c_4e^{\rho_{k1}\omega^2(x-\pi)}$ of (4) when multiplied by $f(t)$ and integrated from 0 to π yield terms of the order M/k, and the term $e^{\rho_{k1}x}$ leads to the first term of (12) plus terms that may be included in g_2, while it is seen from (10) that

$$\frac{1}{D_{kj}} = \frac{1}{\pi} + g_3, \quad |g_3| < \frac{b_9}{k}.$$

The factor b_8, like the other b's similarly used below, is independent of $f(x)$ as well as of k. Similarly,

$$a_{k2} = \frac{1}{\pi}\int_0^\pi f(t)e^{(2/3-2k-2k_0)it}dt + g_4, \quad |g_4| < \frac{Mb_{10}}{k}.$$

Making use of these forms for a_{k1} and a_{k2}, and restricting x to the interval $\epsilon \leqq x \leqq \pi - \epsilon$, so that the asymptotic forms (5) hold, we can write

$$S_n(x) = \sum_{k=1}^{n} \frac{2}{\pi} \int_0^{\pi} f(t) \cos (2k + 2k_0 - 2/3)(x - t)dt$$
$$+ \sum_{k=1}^{n} g_2 e^{(2/3-2k-2k_0)ix} + \sum_{k=1}^{n} g_4 e^{(-2/3+2k+2k_0)ix}$$
$$+ \sum_{k=1}^{n} a_{k1}E_{k1}/k + \sum_{k=1}^{n} a_{k2}E_{k2}/k.$$

Let us denote these five sums by S_{n1}, S_{n2}, S_{n3}, S_{n4}, and S_{n5} respectively and find an upper bound for the derivative of each of them. Differentiating the expression for S_{n2}, we have

$$S'_{n2}(x) = \sum_{k=1}^{n} g_2(2/3 - 2k - 2k_0)ie^{(2/3-2k-2k_0)ix},$$

whence

$$|S'_{n2}(x)| \leqq \sum_{k=1}^{n} \frac{Mb_8}{k} |2/3 - 2k - 2k_0| \leqq nMb_{11}.$$

Similarly

$$|S'_{n3}(x)| \leqq nMb_{12}.$$

It appears from (12) that $|a_{k1}| \leqq b_{13}M$. From this fact, together with the properties of the functions E, it follows that

$$|S'_{n4}(x)| \leqq \sum_{k=1}^{n} |a_{k1}||\overline{E}_{k1}| \leqq \sum_{k=1}^{n} b_{13}Mb_{14} \leqq nMb_{15}.$$

In the same way

$$|S'_{n5}(x)| \leqq nMb_{16}.$$

Now let

$$\sigma_{n2} = \frac{S_{12} + S_{22} + \cdots + S_{n2}}{n};$$

then

$$\sigma'_{n2} = \frac{S'_{12} + S'_{22} + \cdots + S'_{n2}}{n},$$

and hence

$$|\sigma'_{n2}| \leqq \frac{Mb_{11} + 2Mb_{11} + \cdots + nMb_{11}}{n} \leqq nMb_{11}.$$

If we define σ_{n3}, σ_{n4}, and σ_{n5} analogously, then, just as above,

$$|\sigma_{n3}'| \leqq nMb_{12},$$

$$|\sigma_{n4}'| \leqq nMb_{15},$$

$$|\sigma_{n5}'| \leqq nMb_{16}.$$

The arithmetical mean corresponding to S_{n1} may be written in the form

$$\sigma_{n1}(x) = \frac{2}{n\pi}\int_0^\pi f(t)[n\cos(2+2k_0-2/3)(x-t) + (n-1)\cos(4+2k_0 - 2/3)(x-t) + \cdots + \cos(2n+2k_0-2/3)(x-t)]dt$$

$$= \frac{1}{n\pi}\int_0^\pi \frac{f(t)}{\sin^2(x-t)}[\sin^2(n+k_0+2/3)(x-t) - \sin^2(k_0+2/3)(x-t) - n\sin(x-t)\sin(2k_0+1/3)(x-t)]dt.$$

The conditions for differentiation under the integral sign are satisfied (as is seen from the first expression); hence

$$|\sigma_{n1}'(x)| \leqq \frac{M}{n\pi}\left[\int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2(n+k_0+2/3)(x-t)}{\sin^2(x-t)}\right| dt + \int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2(k_0+2/3)(x-t)}{\sin^2(x-t)}\right| dt + n\int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin(2k_0+1/3)(x-t)}{\sin(x-t)}\right| dt\right]. \tag{13}$$

Since x is restricted to the interval $\epsilon \leqq x \leqq \pi-\epsilon$, $x-t$ can take on values only from $\epsilon-\pi$ to $\pi-\epsilon$, and hence the integrand of each of the three integrals of (13) remains finite. Consider the first of these integrals. Let $x-t=u$. Then the integral becomes

$$\int_{z-\pi}^{z}\left|\frac{d}{du}\frac{\sin^2(n+k_0+2/3)u}{\sin^2 u}\right| du$$

which is at most equal to the integral of this same integrand over the interval from $\epsilon-\pi$ to $\pi-\epsilon$. This in turn can be replaced by twice the integral from zero to $\pi-\epsilon$, which represents the total variation of the function

$$\psi(u) = \frac{\sin^2(n+k_0+2/3)u}{\sin^2 u}, \quad 0 < u < \pi,$$

$$\psi(0) = (n+k_0+2/3)^2,$$

in the interval $0 \leqq u \leqq \pi-\epsilon$.

The total variation* of $\psi(u)$ in $(0, \frac{1}{2}\pi)$ cannot exceed $b_{17}n^2$. In $\frac{1}{2}\pi \leqq u \leqq \pi - \epsilon$,

$$\sin u \geqq \sin(\pi - \epsilon), \quad \psi(u) \leqq \frac{1}{\sin^2(\pi - \epsilon)}.$$

By the same reasoning that was used in the case of the interval $(0, \frac{1}{2}\pi)$, $\psi'(u)$ vanishes just once in each subinterval between successive zeros of $\psi(u)$. So the number of maxima and minima of $\psi(u)$ in $(\frac{1}{2}\pi, \pi-\epsilon)$ does not exceed a constant multiple of n, and as $\psi(u)$ is bounded, its total variation does not exceed a quantity of the form $b_{18}n$. The total variation, then, of $\psi(u)$ in $0 \leqq x \leqq \pi - \epsilon$ is not greater than $b_{17}n^2 + b_{18}n$, and so is not greater than $b_{19}n^2$.

The second integral of (13) is independent of n. By changing the variable of integration as we did in the case of the first integral of (13), we find that the absolute value of the second integral cannot exceed the value of

$$2\int_0^{\pi-\epsilon} \left| \frac{d}{du} \frac{\sin^2(k_0 + 2/3)u}{\sin^2 u} \right| du,$$

which may be represented by b_{20}.

Similarly the absolute value of the third term of the bracket in (13) cannot be greater than

$$2n\int_0^{\pi-\epsilon} \left| \frac{d}{du} \frac{\sin(2k_0 + 1/3)u}{\sin u} \right| du,$$

which in turn cannot exceed a quantity of the form $b_{21}n$.

From these inequalities for the integrals in (13) it follows that

$$|\sigma_{n1}'(x)| \leqq \frac{M}{n\pi}[b_{19}n^2 + b_{20} + b_{21}n] \leqq Mnb_{22}.$$

Then since

$$|\sigma_n'(x)| \leqq |\sigma_{n1}'(x)| + |\sigma_{n2}'(x)| + |\sigma_{n3}'(x)| + |\sigma_{n4}'(x)| + |\sigma_{n5}'(x)|,$$

it is seen that

$$|\sigma_n'(x)| \leqq nMb_{22} + nMb_{11} + nMb_{12} + nMb_{15} + nMb_{16},$$

or

$$|\sigma_n'(x)| \leqq nMq_1.$$

When the restriction $a_0 = 0$ is removed, the value of q_1 may conceivably be changed, but the form of the conclusion is the same.

* Cf. E. Carlson, loc. cit., p. 238. The reasoning of the passage cited can be applied directly, with a slight change of notation, since no use was made there of the fact that n was an integer.

Now let $f(x)$ itself be an arbitrary sum of the form (6). Then the coefficients defined by (7) are the same as the given a's for $k \leqq n$, while the corresponding expressions for $k > n$ are equal to zero. If $S_m(x)$ is defined by means of (7) for an arbitrary value of m, and if $\sigma_m(x)$ is the mean of $S_1(x)$, $\cdots$, $S_m(x)$, then $S_m(x)$ is identical with the given $S_n(x)$ when $m \geqq n$, and*

$$S_n(x) = 2\sigma_{2n}(x) - \sigma_n(x).$$

Therefore

$$|S_n'(x)| \leqq 2|\sigma_{2n}'(x)| + |\sigma_n'(x)|,$$

and by application of the lemma

$$|S_n'(x)| \leqq 4nq_1M + nq_1M = nqM.$$

This constitutes a proof of

THEOREM II. *If $|S_n(x)| \leqq M$ in $0 \leqq x \leqq \pi$, then throughout $\epsilon \leqq x \leqq \pi - \epsilon$, $|S_n'(x)| \leqq nqM$, where q is a constant independent of n and of the coefficients in $S_n(x)$, but dependent upon the choice of ϵ.*

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* Cf. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable réelle*, Paris, 1919, pp. 33–34.

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FUNCTIONS OF PLURISEGMENTS*

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Introduction. Vitali has found that for functions of limited variation of a single variable, which we know to correspond to absolutely additive functions of point sets in one dimension, a fundamental concept is that of *elementary discard*. This is a non-decreasing continuous function growing from 0 to 1 in the fundamental interval, but constant through each of a set of intervals whose total measure is the fundamental interval. An absolutely convergent series in terms of these elementary discards builds up the most general continuous function of limited variation whose derivative vanishes almost everywhere. Such a function is indeed merely the continuous case of what is called a *discard*—which in general measures the quantity by which a function of limited variation falls short of being absolutely continuous.†

That these results apply to more than one dimension is not obvious, on account of the variety possible in the function of singularities. Indeed, the instrument which is adapted for the analysis of the general discard in the typical case of two dimensions is not the point function, but Volterra's function of curves—more particularly, the *function of plurisegments*. Like the point function, it yields an analysis in terms of closed sets; unlike the point function, it does not prefer discontinuities distributed along a vertical or horizontal line and is therefore a more proper complement of the absolutely additive function of point sets.‡

An analysis in terms of point functions has been made§ by isolating denumerable sets of lines of discontinuity parallel to the axes. Such a treatment, however, is not as general as this one since in a two-dimensional problem discontinuities located on lines not parallel to the axes or on other curves are equally important.

1. Preliminary notions

The fundamental rectangle shall be denoted by Δ. A segment of Δ is a rectangle contained in Δ with its edges parallel to those of Δ.

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† G. Vitali, *Analisi delle funzioni a variazione limitata*, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 388.

‡ Compare the pages devoted to functions of two variables in de la Vallée Poussin, *Sur l'intégrale de Lebesgue*, these Transactions, vol. 16 (1915). See also de la Vallée Pousin, *Intégrales de Lebesgue*, Paris, 1916, p. 90.

§ H. Lebesgue, Annales de l'Ecole Normale Supérieure, 1910, p. 361.

Two segments of Δ are said to be distinct when they have no common internal points; they can, however, have portions of their boundaries common.

A segment s_1 is contained in another segment s_2 when every point of s_1 belongs to s_2; we do not exclude the case where the boundaries have points in common. Similarly, s_1 is contained in a finite number of segments, $s^{(1)}, \cdots, s^{(k)}$, if every point of s_1 belongs to some $s^{(i)}$.

The ensemble of a finite or denumerable infinity of segments, two by two distinct, shall be called a plurisegment. The plurisegment composed of a finite number of segments is said to be finite, otherwise infinite.

A point internal to any segment of a plurisegment is said to be internal to the plurisegment.

If p_1 and p_2 are two plurisegments, then p_2 is said to be contained in p_1 if every segment of p_2 is contained in a finite number of segments of p_1.

If p_1 and p_2 are two plurisegments and every segment of p_1 is distinct from every segment of p_2 then p_1 and p_2 are said to be distinct.

DEFINITION.* Let s_1 and s_2 be two boundaries of segments in Δ exterior to each other except for a common portion s', and let s_3 be the curve composed of s_1 and s_2 with the omission of s'; if for all such segments the relation

$$F(s_1)+F(s_2)=F(s_3)$$

is satisfied, then $F(s)$ is said to be an additive function of segments.

DEFINITION. If p is an infinite plurisegment in Δ and if the relation

$$F(p)=\sum_1^\infty{}_i F(s_i) \tag{1}$$

holds for all such plurisegments, then $F(p)$ is said to be an absolutely additive function of segments, or of plurisegments.

We shall assume that $F(p)$ is finite on every plurisegment in Δ. It is an immediate consequence of this assumption that the series (1) is absolutely convergent.

THEOREM A. *An absolutely additive function $F(p)$ of segments is bounded for all plurisegments in Δ.*

We say that $F(p)$ is *bounded on* Δ if it is bounded for all plurisegments in Δ.

Suppose $F(p)$ is not bounded on Δ. Then a finite plurisegment p_1 can be found in Δ on which F is as large as we please, i. e., $|F(p_1)|>1+|F(\Delta)|$

* Volterra, *Les Fonctions de Lignes*, Paris, 1913.

and, therefore, if $\bar{p}_1$ is the complementary plurisegment such that $p_1+\bar{p}_1=\Delta$ then $|F(p_1)|>1$, $|F(\bar{p}_1)|>1$, since $F(p_1)+F(\bar{p}_1)=F(\Delta)$.

Now F is unbounded in p_1 or $\bar{p}_1$, suppose in $\bar{p}_1$. Then we can find a finite plurisegment p_2 in $\bar{p}_1$ in which F is as large as we please, say $|F(p_2)|>1+|F(\bar{p}_1)|$; let $\bar{p}_2$ be the complement of p_2 in $\bar{p}_1$; then $|F(p_2)|>1$, $|F(\bar{p}_2)|>1$ and F is unbounded, say in $\bar{p}_2$.

We thus get a sequence of distinct finite plurisegments $p_1, p_2, \cdots$, in each of which $|F|>1$.

Hence the series

$$|F(p_1)|+|F(p_2)|+\cdots$$

is divergent. This contradicts our statement that F is an absolutely additive function of segments. Hence F is bounded on Δ.

DEFINITION. A function $F(p)$ of plurisegments, i. e., an absolutely additive function of segments, is said to be absolutely continuous if for any positive quantity ϵ, assigned in advance, a quantity $\delta>0$ can be found such that $|F(p)|<\epsilon$ for every plurisegment p whose measure is less than δ.

THEOREM B. *An absolutely additive function of segments is the difference of two non-negative functions of the same nature.*

Consider any plurisegment p in Δ, and let $U(p)$ be the upper bound of $F(p)$ for all finite plurisegments in p. Then $U(p)$ is non-negative since in any segment of p a segment can be found for which F is arbitrarily small. $U(p)$ is finite, since F is bounded on Δ, by Theorem A. If $F(p)$ were negative for every plurisegment we should write $F(p)=-N(p)$. In general write $N(p)=U(p)-F(p)\geqq 0$.

We have to show now that $U(p)$ is absolutely additive. Suppose first that p consists of two segments s_1 and s_2; then

$$U(p)\geqq U(s_1)+U(s_2)\,,$$

since any finite plurisegment in s_1 plus any finite plurisegment in s_2 is a finite plurisegment in p.

Now we can find a finite plurisegment in p, call it p_ϵ, for which

$$F(p_\epsilon)>U(p)-\epsilon \text{ for any } \epsilon>0\,,$$

and this plurisegment will be divided into two finite plurisegments p_{ϵ_1}, p_{ϵ_2}, contained in s_1 and s_2, such that

$$F(p_{\epsilon_1})+F(p_{\epsilon_2})>U(p)-\epsilon$$

for any $\epsilon>0$ and therefore

$$U(s_1)+U(s_2)\geqq U(p)\,.$$

It follows that

$$U(s_1)+U(s_2)=U(p)\ .$$

If p is composed of n segments it is easily seen that

$$U(p)=\sum_1^n{}_i\ U(s_i)\ .$$

Now suppose that p is an infinite plurisegment, consisting of the segments $s_1, s_2, \cdots$; we have

$$U(p)\geqq\sum_1^n U(s_i) \text{ for any } n$$

and therefore $U(p)\geqq\sum_{i=1}^{\infty}U(s_i)$, the latter quantity being finite, $\leqq U(\Delta)$.

On the other hand a finite plurisegment $\bar{p}$ can be found, contained in p, such that for any ϵ given in advance

$$F(\bar{p})>U(p)-\epsilon\ .$$

But by definition $\bar{p}$ is contained in a finite plurisegment $p^{(m)}$ whose segments $s_{n_1}, s_{n_2}, \cdots, s_{n_m}$ are segments of p. Hence, as we have shown in the case where p is finite,

$$U(s_{n_1})+U(s_{n_2})+\cdots+U(s_{n_m})=U(p^m)\geqq U(\bar{p})\geqq F(\bar{p})\ ,$$

and therefore

$$\sum_1^{\infty}{}_i\ U(s_i)>U(p)-\epsilon\ ,$$

i. e.,

$$\sum_1^{\infty}{}_i\ U(s_i)\geqq U(p)\ .$$

We have thus shown that $\sum_{i=1}^{\infty}U(s_i)=U(p)$.

Let us write

$$N(p)=U(p)-F(p)\geqq 0\ .$$

The desired decomposition is thereby effected.

COROLLARY. *If $F(p)$ is absolutely continuous then $U(p)$ is absolutely continuous.*

It is evident from the above that we do not lose any generality if we restrict our discussion to non-negative functions of plurisegments in the sequel.

DEFINITION. Consider a sequence of positive numbers $\sigma_1>\sigma_2>\sigma_3>\cdots$ such that $\lim_{n\to\infty}\sigma_n=0$ and a plurisegment p in Δ. Consider all the finite

plurisegments M_{σ_n}, contained in p, of measure $\leqq\sigma_n$. Let ρ_n be the upper bound of $F(M_{\sigma_n})$ for all M_{σ_n}. Then $\rho_1\geqq\rho_2\geqq\cdots$. Let ρ be the lower limit of ρ_n. Then ρ is called the discard of F on p, and is denoted by the symbol S_pF. It is evident that ρ is independent of the choice of the sequence $\{\sigma_i\}$.

2. Significant properties of discards

THEOREM C. *A necessary and sufficient condition that a non-negative F defined in Δ have a zero discard in Δ is that F be absolutely continuous in Δ.*

The proof is immediate from the above definitions.

THEOREM. *The discard of F is an additive function of segments, i. e.,*

$$S_pF=\sum_1^n{}_i\, S_{s_i}F\ .$$

For convenience consider a plurisegment p composed of two segments s_1 and s_2. The sum of any pair p_σ and $p_{\bar\sigma}$ of finite plurisegments in s_1 and s_2 respectively is a finite plurisegment in p and therefore

$$S_pF\geqq S_{s_1}F+S_{s_2}F\ .$$

On the other hand if S_pF is the discard of F in p a finite plurisegment p_s in p can be found such that

$$F(p_s)>S_pF-\epsilon\ ,$$

for any $\epsilon>0$ and no matter how small the measure of p_s. This plurisegment can be split into two parts p_s' and p_s'' contained respectively in s_1 and s_2 such that

$$F(p_s')+F(p_s'')>S_pF-\epsilon\ .$$

It follows that

$$S_{s_1}F+S_{s_2}F=S_pF\ .$$

Our theorem is therefore demonstrated. We may proceed similarly if p is composed of n segments.

THEOREM. *The discard of $F(p)$ is additive for any plurisegment, i. e.,*

$$S_pF(p)=\sum_1^\infty{}_i\, S_{s_i}F$$

where p is a plurisegment in Δ.

We have

$$S_pF\geqq\sum_1^n{}_i\, S_{s_i}F$$

for $S_pF \geqq \sum_{i=1}^{n} S_{s_i}F$, since any finite plurisegment in $(s_1+s_2+\cdots+s_n)$ is contained in p. Since $S_{s_i}F \leqq F(s_i)$, we can take n so large that

$$\sum_1^n{}_i\, S_{s_i}F > \sum_1^\infty{}_i\, S_{s_i}F-\epsilon \text{ for any } \epsilon>0\ ,$$

or

$$S_pF \geqq \sum_1^\infty{}_i\, S_{s_i}F\ .$$

On the other hand we can take n so large that

$$\sum_{n+1}^\infty{}_i\, S_{s_i}F<\frac{\epsilon}{3} \text{ and } \sum_{n+1}^\infty{}_i\, F(s_i)<\frac{\epsilon}{3}\ .$$

Now take σ so small that for each of the segments s_i and any finite plurisegment p_i of measure $\leqq\sigma$ contained in s_i $(i=1, 2, \cdots, n)$

$$F(p_i)<S_{s_i}F+\frac{\epsilon}{3n} \text{ for any } \epsilon>0\ .$$

Now any finite plurisegment p' contained in p of measure $\leqq\sigma$ is divided into two plurisegments p_1' and p_2' where p_1' is in $(s_1+s_2+\cdots+s_n)$ and p_2' is in $(s_{n+1}+\cdots)$. In fact, each part of a segment of p' which is not contained in $(s_1+s_2+\cdots+s_n)$ is contained in a finite number of segments of $(s_{n+1}+\cdots)$.

Then certainly

$$F(p_1') < \sum_1^n{}_i\, S_{s_i}F+\frac{\epsilon}{3}\ ,$$

and

$$F(p_2')<\frac{\epsilon}{3}\ ,$$

therefore

$$F(p')=F(p_1')+F(p_2') < \sum_1^n{}_i\, S_{s_i}F+\frac{2\epsilon}{3} < \sum_1^\infty{}_i\, S_{s_i}F+\epsilon\ .$$

Hence

$$S_pF \leqq \sum_1^\infty{}_i\, S_{s_i}F\ .$$

Hence our theorem follows.

THEOREM D. *The function SF is its own discard, i. e., $S_pF=S_p[SF]$ where p is any plurisegment in Δ.*

First let us note that SF is non-negative as is evident from our definition of discard.

Let $S_pF=k$ and $S_p[SF]=k_1$; then $k_1 \leqq k$.

Put $t=k-k_1$. There exists a $\sigma>0$ such that in every finite plurisegment p_σ in p of measure $\leqq \sigma$, $SF<k_1+t/2$.

On the other hand, no matter how small σ happens to be, there is a finite plurisegment p_σ in p for which $F(p_\sigma)>k-\epsilon$ where ϵ is given in advance; in particular, then, where $F(p_\sigma)>k-t/2$. Hence $S_{p_\sigma}F \geqq k-t/2$. But this is a contradiction, since it has just been shown that SF on p_σ is $<k_1+t/2$.

THEOREM E. *If g and φ are two non-negative absolutely additive functions of segments in Δ then if $F=g+\varphi$ we have $S_pF=S_pG+S_p\varphi$ where p is any plurisegment in Δ.*

Given $\epsilon>0$, $\sigma>0$ and arbitrary, there exists in p a finite plurisegment p_1 of measure $<\sigma/2$ in which $g>S_pg-\epsilon/2$ and a finite plurisegment p_2 of measure $<\sigma/2$ in which $\varphi>S_p\varphi-\epsilon/2$. In the logical sum of p_1 and p_2 we have, a fortiori,

$$F=g+\varphi>S_pg+S_p\varphi-\epsilon\ ,$$

therefore

$$S_pF \geqq S_pg+S_p\varphi\ .$$

Now for every $\epsilon>0$, a $\sigma>0$ exists such that in every finite plurisegment in p of measure $\leqq \sigma$ we have $g<S_pg+\epsilon/2$ and $\varphi<S_p\varphi+\epsilon/2$; therefore for every finite plurisegment of measure $\leqq \sigma$ we have

$$F=g+\varphi<S_pg+S_p\varphi+\epsilon$$

for any $\epsilon>0$. Therefore

$$S_pF \leqq S_pg+S_p\varphi\ .$$

This together with the previous inequality proves the theorem.

COROLLARY. *If g and φ are non-negative functions of plurisegments in Δ and if $F(p)=g(p)-\varphi(p)$ is non-negative, then $S_pF=S_pg-S_p\varphi$.*

This is easily seen to be the case since

$$S_pF+S_p\varphi=S_pg\ .$$

Write

$$\nabla F(p)=F(p)-S_pF\ .$$

By Theorem E we have $S_pF=S_p\nabla F+S_p[SF]$; but $S_p[SF]=S_pF$ by Theorem D; therefore $S_p\nabla F=0$, i. e., ∇F is absolutely continuous in Δ, by Theorem C.

We can therefore state that *every non-negative function of plurisegments is the sum of a discard and an absolutely continuous function of plurisegments.*

3. Point values and continuous discards

Definition. Let $U(p)$ be a non-negative function of plurisegments of Δ, and A a point of Δ. Define $U_{++}(A)$ as $\lim_{d\to 0} U(s_{++})$ where s_{++} is a segment in Δ whose lower left hand vertex is A and whose diameter d is made to approach zero. If for A there is no s_{++} in Δ we define $U_{++}(A)$ as 0. Similarly, define $U_{+-}(A)$ corresponding to segments on the lower right hand of A, etc. Then $U(p)$ is said to have point values at A if not all the $U_{++}(A)$, $U_{+-}(A)$ etc. are zero; moreover, we write

$$U(A)=U_{++}(A)+U_{+-}(A)+U_{-+}(A)+U_{--}(A)\ .$$

If A happens to be a boundary point of Δ two or three terms in the above sum will vanish.

Obviously there cannot be more than a denumerable infinity of such points A.

Having defined a point value of $U(p)$ we may define now a function of point values for plurisegments p. Let e be the denumerable set of points A at which $U(p)$ has point values. We define $f(p)$, the corresponding function of point values, by the equation

$$f(p)=\sum_e\{U_{++}(A)+U_{+-}(A)+U_{-+}(A)+U_{--}(A)\}\ ,$$

where only those terms are included in the summation for which the corresponding segments s_{++} or s_{+-} etc. are in the limiting process ultimately contained in p. The $f(p)$ is in fact an absolutely additive function of plurisegments.

It is verified directly that the function $U(p)-f(p)$ has no point values, and is positive or zero.

In general for $F(p)=U(p)-N(p)$ we define

$$F(A)=U(A)-N(A)\ ,\quad F_{++}(A)=U_{++}(A)-N_{++}(A)\ ,\ \text{etc}\ .$$

Further we name the function

$$f_F(p)=f_U(p)-f_N(p)$$

the function of point values for F.

It follows that the function $F_1(p)=F(p)-f_F(p)$ has no point values.

If F, non-negative in Δ, is without point values then SF is a function without point values, since

$$SF=F-\nabla F\ .$$

Definition. If $F(p)$ is non-negative in Δ and z is an interior point of a segment for which $F=0$, then z is a point of invariability. Any point not a point of invariability is a point of variability.

DEFINITION. Let φ be a discard in Δ, where φ has no point values and $\varphi(\Delta)=1$. Let the points of variability of φ constitute a set of measure zero. Such a discard shall be called an *elementary discard* in Δ.

THEOREM F. *The points of variability of $F(p)$ non-negative in Δ, without point values, constitute a perfect set G.*

If z is a point of variability, given any neighborhood arbitrarily small of z, we can find a segment s in that neighborhood of which z is an interior point, and for which $F(s)\neq 0$. Also since F is a non-negative function of curves it must be >0 for any segment which has a point of variability as an interior point.

Let z be a limit point of G and let s be any segment of which z is interior; then some points of G occur also as interior points of s. Therefore $F(s)\neq 0$ and z is a point of G.

There is a segment s' for which z is an interior point for which $F(s')=\eta>0$. If z is an isolated point of variability, s' may be taken small enough so that there are no other points of variability in s'. There is a rectangle s'' contained in s' small enough (and with z as an interior point) for which $F(s'')<\eta/2$. Now in or on the boundary of the region σ included between s' and s'' there is no point of variability. Hence each point is an interior point of a segment for which $F=0$, and we can cover this region with a finite number* n of rectangles s_i on each of which $F(s_i)=0$. But σ is a finite plurisegment and $F(\sigma)\leqq\sum_{i=1}^{n}F(s_i)=0$. Hence $F(s')\leqq\eta/2$. We are thus led to a contradiction, and consequently z cannot be a point of variability. Therefore G is perfect.

THEOREM. *If $\varphi_1(p)$, $\varphi_2(p)$, $\varphi_3(p)$, $\cdots$ is a series of elementary discards in Δ and $\sum_i k_i$ is a convergent series of positive terms, then $\sum_i k_i\varphi_i(p)$ converges uniformly and towards a function $\psi(p)$ which is a discard and has no point values.*

First it is obvious that $\sum_i k_i\varphi_i(p)$ converges uniformly.

If the k_i are merely finite in number the result is obvious from Theorem E.

If the k are infinite in number then

$$S_p\psi \geqq \sum_1^r{}_i\, k_i,\ S_p\varphi_i = \sum_i k_i\varphi_i(p)$$

for any r, consequently

$$S_p\psi \geqq \sum_1^\infty{}_i\, k_i\varphi_i(p)=\psi(p)\ .$$

* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 13.

But

$$S_p\psi \leqq \psi(p) .$$

Therefore

$$\psi(p) = S_p\psi ,$$

i. e., $\psi(p)$ is a discard. It is without point values since $\sum_i k_i\varphi_i(p)$ converges uniformly.

THEOREM G. *If $\psi(p)$ is a discard, without point values, in Δ, then $\psi(p)$ is equal to a series of the form $\sum_i k_i\varphi_i(p)$ with a finite or infinite number of terms, in which the $\varphi_i(p)$ are elementary discards in Δ and the k_i are positive constants for which $\sum_i k_i$ is convergent.*

Given $\epsilon>0$ and arbitrarily small but $<\psi(\Delta)$ and $\sigma>0$; then a finite plurisegment p_1 can be found in Δ of measure $<\sigma/2$ in which $\psi>\psi(\Delta)-\epsilon/2$. It is possible to find in p_1 a finite plurisegment, of measure $<\sigma/4$, in which

$$\psi>\psi(\Delta)-\frac{\epsilon}{2}-\frac{\epsilon}{4} , \text{ etc} .$$

The points common to all the plurisegments form a set G of measure zero. This set is closed since $p_1, p_2, \cdots$ are closed sets.*

Let us consider a finite plurisegment p which contains all the points of G in its interior. After a certain point, say i, the plurisegments $p_{i+1}, p_{i+2}, \cdots$ are contained in p. If this were not true for every p_{i+n} there would be an infinite sequence among the sets p_{i+n} for which the statement is not true, say $p_{k_1}, p_{k_2}, \cdots$. Now insert in p_{k_i} the vertices of p internal to it. We have p_{k_i} divided into two parts, the first consisting of a finite plurisegment p'_{k_i} contained in p and the second p''_{k_i} distinct from p. The plurisegments

$$p''_{k_1} , p''_{k_2} , \cdots$$

are each contained in the preceding and hence have points in common. These points are not internal to p and on that account do not belong to G, which is contrary to the definition of G.

The discard of ψ in p is therefore

$$\geqq\psi(\Delta)-\epsilon(\tfrac{1}{2}+\tfrac{1}{4}+\cdots)=\psi(\Delta)-\epsilon .$$

If s is any segment, let G_s be the subset of G in s, or on its boundary. As a special case G_s may be a null set. By the preceding analysis the discard of $\psi(s)$ in any finite plurisegment of which the points of G_s, if any, are interior points (with respect to s as fundamental segment) is $\geqq \psi(s)-\epsilon$.

* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 11.

Let $l(s)$ be the lower limit of the discard of $\psi(s)$ for all such finite plurisegments contained in s. The value of $l(s)$ is zero if no points of G are contained in s.

If p_0 is any finite plurisegment composed of segments $s_i (i=1, 2, \cdots, n)$, let G_{p_0} be the subset of G in p_0 or on its boundary. By the preceding analysis the value of $\psi(p)$ in any finite plurisegment p' of which the points of G_{p_0} are interior points (with respect to the points of p_0 as a fundamental set) is $\geqq \psi(p_0) - \epsilon$. Let $l(p_0)$ be the lower limit of $\psi(p')$ for all such finite plurisegments p' contained in p_0. We wish to show that $l(p_0) = \sum_{i=1}^{n} l(s_i)$.

In fact any n finite plurisegments p_i' of the kind mentioned above, one contained in each s_i, form a finite plurisegment contained in p_0, for which all the points of G_{p_0} are interior points (with respect to p_0 as a fundamental set). Hence

$$l(p_0) \leqq \sum l(s_i) .$$

On the other hand any finite plurisegment p' contained in p_0 which contains all the points of G_{p_0} as interior points with respect to p_0 (as a fundamental set) is divided by the lines of p_0 into a finite number $\leqq n$ of finite plurisegments p_i' each of which contains all the points of G_{s_i} as interior points with respect to s_i (as fundamental set). Hence

$$\psi(p') = \sum_i \psi(p_i') .$$

In particular let p' be a plurisegment of this kind for which

$$\psi(p') - l(p_0) \leqq \epsilon .$$

But $\sum \psi(p_i') \geqq \sum l(s_i)$, and therefore

$$l(p_0) + \epsilon \geqq \sum \psi(p_i') \geqq \sum l(s_i) ,$$

whatever ϵ. In other words

$$l(p_0) \geqq \sum_i l(s_i) .$$

Hence with the help of the previous inequality,

$$l(p_0) = \sum_i l(s_i) .$$

On account of the fact that $l(p) \leqq \psi(p)$ for any finite plurisegment, the additive function of segments defines $l(p)$ as an absolutely additive function of plurisegments.

Since $l(s) = 0$ for any segment which does not contain points of G, the points of variability of that function of plurisegments form a perfect set,

Theorem F, of measure zero. Therefore $l(p)/l(\Delta)$ is an elementary discard, by definition.

Let us rewrite $l(p)/l(\Delta)$ as $\varphi_1(p)$ and define $k_1=l(\Delta)$; then

$$\psi(p)-k_1\varphi_1(p)\leqq\epsilon$$

for any plurisegment of Δ. In particular

$$\psi(\Delta)-k_1\varphi_1(\Delta)\leqq\epsilon\ .$$

Denote $\psi(p)-k_1\varphi_1(p)$ by $\psi_1(p)$.

If $\psi_1(\Delta)=0$ then

$$\psi(p)=k_1\varphi_1(p)\ ,$$

and the theorem is proved.

If $\psi_1(\Delta)>0$ then take $\epsilon_1<\psi_1(\Delta)$ and $<\epsilon/2$ but >0; then we can find an elementary discard φ_2 and a constant $k_2>0$ for which

$$\psi_2(p)=\psi_1(p)-k_2\varphi_2(p)$$

is a discard without point values and where $\psi_2(\Delta)<\epsilon_1$. If $\psi_2(\Delta)=0$, then

$$\psi(p)=k_1\varphi_1(p)+k_2\varphi_2(p)$$

and the theorem is proved.

If $\psi_2(\Delta)>0$ we continue the process and obtain $\varphi_r(p)$ and $\psi_r(p)$. If, for every r, $\psi_r(p)>0$, then for every r

$$\psi_r(\Delta)<\frac{\epsilon}{2^{r-1}}\ ,$$

and therefore for every p

$$\lim_{r\to\infty}\psi_r(p)=0\ .$$

But

$$\psi_r(p)=\psi(p)-\sum_1^r{}_i\, k_i\varphi_i(p)$$

and therefore

$$\psi(p)=\sum_1^\infty{}_i\, k_i\varphi_i(p).$$

From the definition of k_i it is clear that $\sum_{i=1}^\infty k_i$ is convergent.

4. Derivatives and decomposition

Definition. The derivative of $F(p)$ at z is defined as

$$\lim_{m(p)\to 0}\frac{F(p)}{m(p)}$$

where p is a plurisegment, $m(p)$ is the measure of p and p is subjected to the condition that it be a member of a regular family with respect to z. We define regular family as follows:

The ratio of $m(p)$ to that of the smallest square segment, with sides parallel to those of Δ, of center z which contains p does not tend to zero with $m(p)$, when p belongs to this family.

DEFINITION. Consider a sequence of positive numbers, h, $h/2$, $h/4$, $h/8$, $\cdots$, which are half sides of squares of center P, in Δ. The restricted upper derivative of a non-negative absolutely additive function of plurisegments $\varphi(p)$* at P is defined as

$$\overline{\lim_{m\omega\to 0}} \frac{\varphi(\omega_n)}{m\omega_n} = \bar{\bar{D}}\varphi ,$$

where ω_n is a square of half side $h/2^n$ and center at P.

LEMMA (a). *The $\bar{D}\varphi$, defined as above, is measurable* (B).†

Let

$$\frac{\varphi(\omega_n)}{m\omega_n} = \frac{F(x,y\,|\,h/2^n)}{4(h/2^n)^2} ,$$

where (x, y) are the coördinates of the point P at which $\bar{D}\varphi$ is calculated and $h/2^n$ is the half side of a square of center P. For a fixed n, $F(x, y|h/2^n)$ is a function of (x, y). We now put

$$\begin{aligned} F(x,y\,|\,h/2^n) = {} & g(x+h/2^n\ ,\ y+h/2^n) - g(x+h/2^n\ ,\ y-h/2^n) \\ & + g(x-h/2^n\ ,\ y-h/2^n) - g(x-h/2^n\ ,\ y+h/2^n) , \end{aligned}$$

where we define $g(x\pm h/2^n,\ y\pm h/2^n) = \varphi(\omega)$, ω being the rectangle with vertices $(0, 0)$, $(0, y\pm h/2^n)$, $(x\pm h/2^n, 0)$, $(x\pm h/2^n, y\pm h/2^n)$.

In order to prove $F(x, y|h/2^n)$ measurable (B) as a function of (x, y), it will be sufficient to show that $g(x+h/2^n, y+h/2^n)$ is measurable (B) (n being held fast). Given any number $c \geqq 0$. Then the set E where $g \leqq c$ is obviously measurable (B). In fact E is a closed set minus a set which is of zero measure (B).

We conclude that $F(x, y|h/2^n)/4(h/2^n)^2$ is measurable (B) and therefore $\bar{D}\varphi$ is measurable (B).‡

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* In order to take care of boundary points, by definition $\varphi(p) = 0$ for any plurisegment p not overlapping Δ.

† We shall write measurable (B) to denote measurability in the Borel sense.

‡ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 30.

LEMMA (b). *If $\bar{\bar{D}}\varphi$ is zero almost everywhere in Δ, then $D\varphi$, calculated for any regular family, is zero almost everywhere in Δ.*

It is easily seen that

$$\frac{4\varphi(\omega_n)}{\left(\frac{2h}{2^n}\right)^2} \geqq \frac{\varphi(\bar{\omega}_n)}{\delta^2},$$

where δ^2 is the measure of any square $\bar{\omega}_n$, with center P, and included between the squares of half sides $h/2^n$ and $h/2^{n+1}$.

Given $\epsilon > 0$ and arbitrary. If $\bar{D}\varphi$ is the symmetric upper derivative of φ at P, then a square $\bar{\omega}_n$ can be found such that

$$\frac{\varphi(\omega_n)}{\delta^2} > \bar{D}\varphi - \epsilon,$$

this inequality holding for an infinity of squares of measure $< \delta^2$. Now for any $n \geqq n_0$

$$4\bar{\bar{D}}\varphi + \eta > \frac{4\varphi(\omega_n)}{(2h/2^n)^2} \geqq \frac{\varphi(\bar{\omega}_n)}{\delta^2},$$

$\eta > 0$ and arbitrary, since we choose n so that

$$\left(\frac{2h}{2^{n+1}}\right)^2 \leqq \delta^2 \leqq \left(\frac{2h}{2^n}\right)^2.$$

It follows that

$$4\bar{\bar{D}}\varphi + \eta > \bar{D}\varphi - \epsilon,$$

and since $\epsilon > 0$, $\eta > 0$ are arbitrary, we have

$$4\bar{\bar{D}} \geqq \bar{D}\varphi \geqq \underline{D} \geqq 0.$$

If $\bar{D}\varphi = 0$ almost everywhere in Δ, then $\bar{D}\varphi = \underline{D}\varphi = 0$ almost everywhere in Δ. Hence,* as is well known, $D\varphi = 0$ almost everywhere in Δ for any choice of regular families serving to define it.

THEOREM. *The non-negative function $f(p)$ of point values of $F(p)$, an absolutely non-negative function of plurisegments, has a zero derivative almost everywhere in Δ. The derivative is zero independently of its definition.*

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* de la Vallée Poussin, *Intégrale de Lebesgue*, 1916, p. 60.

Consider the function

$$f(\omega)=\sum_i\{U_{++}(A_i)+U_{+-}(A_i)+U_{-+}(A_i)+U_{--}(A_i)\} \tag{1}$$

(defined at the beginning of § 3), where ω is any rectangle in Δ. Since the series (1) is convergent, an i exists such that

$$R_i(\Delta)=\sum_{i\geq i_0}^{\infty}\{U_{++}(A_i)+U_{+-}(A_i)+U_{-+}(A_i)+U_{--}(A_i)\}<\epsilon\ ,$$

$\epsilon>0$ and arbitrary.

We may write

$$f(\omega)=f_i(\omega)+R_i(\omega) \qquad (i\geqq i_0).$$

Evidently $\overline{D}f_i=0$ almost everywhere in Δ. If $\overline{D}R_i$ is not almost everywhere zero in Δ, then, since it is measurable, Lemma (a), there must be a set E_i, of measure $\sigma>0$, in Δ where it is $>\delta>0$ and where σ is independent of i.

Every point of E_i is the center of an infinity of squares ω_k, of measure as small as we please, and on each of which $R_i(\omega_k)/m\omega_k>\delta$. By Vitali's Lemma,* we can cover E_i by a finite or denumerable infinity of non-overlapping squares selected from the ω_k such that their measure $\geqq\sigma$ and differs from σ by as little as we please. Then if $\omega=\sum\omega_k$ we shall have

$$R_i(\omega)>\delta\sigma$$

where $R_i(\omega)=\sum_k R_i(\omega_k)\leqq R_i(\Delta)<\epsilon$, arbitrarily small. Hence $\epsilon>\delta\sigma$, but since $\delta\sigma$ is independent of ϵ, this cannot be true. We are thus led to a contradiction and, therefore, $\overline{D}f=0$ almost everywhere in Δ. We see from Lemma (b) that $Df=0$ almost everywhere in Δ, where Df is a derivative defined on any regular family.

THEOREM. *A continuous non-negative discard $\psi(p)$ has a zero derivative almost everywhere in Δ. The derivative is zero independently of its definition.*

By Theorem G of § 3 we can write

$$\psi_n(\omega)=\psi(\omega)-\sum_1^n{}_i\, k_i\varphi_i(\omega)\ ,$$

where ω is any rectangle in Δ and $\varphi_i(\omega)$ are elementary discards in Δ. We can choose n_0 so large that $\psi_n(\Delta)<\epsilon$ for any $n\geqq n_0$, $\epsilon>0$ being arbitrary and preassigned. The term $\sum_{i=1}^n k_i\varphi_i(\omega)$ has a zero derivative almost everywhere in Δ, since the set of points of variability of $\sum_{i=1}^n k_i\varphi_i(\omega)$ which we write as $(G_1+G_2+\cdots+G_n)$ is perfect and of measure zero. The remainder of this proof is, mutatis mutandis, the same as above.

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*H. Lebesgue, Annales de l'Ecole Normale Supérieure, 1910, p. 391.

COROLLARY. *Any discard has a zero derivative almost everywhere in* Δ.

In fact any discard can be written as a continuous discard plus its function of point values.

THEOREM. *A non-negative absolutely additive function of plurisegments,* $F(p)$, *in* Δ *may be written as follows*:

$$F(p)=S_pF+\int_p DF\,d\sigma\ .$$

This is evident since $F(p)=S_pF+A(p)$, where $A(p)$ is absolutely continuous and $DS_pF=0$ almost everywhere in Δ.*

THEOREM H. *The decomposition of an absolutely additive non-negative function of plurisegments,* $F(p)$, *into a function whose derivative* $=0$ *almost everywhere in* Δ *and an absolutely continuous function of plurisegments, is unique.*

Suppose

$$F(p)=G(p)+A(p)$$

where $DG=0$ almost everywhere in Δ and $A(p)$ is absolutely continuous. We see that

$$F(p)=G(p)+\int_p DF\,d\sigma\ .$$

It follows that

$$G(p)=S_pF \text{ and } \int_p DF\,d\sigma=A(p)\ .$$

5. ABSOLUTELY ADDITIVE FUNCTIONS OF SETS MEASURABLE BOREL

Consider

$$F(\omega)=S_\omega F+\int_\omega DF\,d\sigma\ ,$$

where $F(p)$ is an absolutely additive non-negative function of plurisegments in Δ.

Now† to $F(p)$ there corresponds one and only one absolutely additive function, $f(e)$, of point sets measurable (B) which coincides with $F(p)$ on every rectangle of continuity (i. e., $f(e)=F(\omega)$ whenever $f(s)=0$, e being the

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* The existence and summability of $DA(p)$ can be proved by exactly the same reasoning as that employed by Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918.

† G. C. Evans, The Rice Institute Pamphlet, vol. 7 (Oct., 1920), p. 271.

set of points which constitutes the rectangle ω and s being the boundary of ω). Let $f(e)$, $S_e f$, $a(e)$ be the functions of point sets measurable (B) which correspond to

$$F(p),\ S_p F,\ \int_p DF d\sigma$$

respectively, these being non-negative functions of point sets. In fact $f(e)$ is the lower limit of $F(p)$ for all plurisegments p which constitute open sets containing e.*

THEOREM I. *If $f(e)$ is the function of sets, measurable* (B), *which corresponds to $F(p)$, an absolutely additive non-negative function of plurisegments, then*

$$f(e) = S_e f + \int_e Df\, d\sigma\ ;$$

$S_e f$ is the function of sets defined by $S_p F$, and DF is a derivative of $F(e)$ calculated for any regular family.

We have, from the definition of $f(e)$,

$$F(p) < f(e) + \epsilon\ , \qquad \epsilon > 0 \text{ and arbitrary.} \tag{1}$$

The above inequality holds for an infinity of plurisegments p constituting open sets which contain e, e being the set of points which constitutes a closed rectangle ω. Since

$$S_p F \geqq S_e f\ , \qquad \int_p DF\, d\sigma \geqq a(e)$$

hold for any open p containing e and in particular for a p for which (1) holds, we have

$$a(e) + S_e f \leqq f(e)\ . \tag{α}$$

On the other hand, given $\epsilon > 0$ and arbitrary, we have

$$S_p F < S_e f + \frac{\epsilon}{2}\ , \qquad \int_p DF\, d\sigma < a(e) + \frac{\epsilon}{2}\ ,$$

these relations holding at the same time for an infinitude of open p which contain e. In fact the second inequality has place for all p which are contained in p_0 and contain e if only $m(p_0 - e)$ is sufficiently small. We conclude that

$$f(e) \leqq S_e f + a(e)\ . \tag{β}$$

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* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 86 ff.

Combining the two inequalities, (α) and (β), we see that

$$f(e)=S_e f+a(e)$$

on any rectangle in Δ, therefore on any set measurable (B) in Δ.

It is easily seen that $a(e)$ is absolutely continuous as a function of point sets measurable (B). We also see, from the definition of $S_e f$, that $DS_e f=0$ almost everywhere in Δ, independently of the choice of regular families serving to define $DS_e f$. We may, therefore, write

$$f(e)=S_e f+\int_e Df\,de\,,$$

since $Da(e)$ exists almost everywhere in Δ, is summable* and is equal to Df almost everywhere in Δ; Df is defined, arbitrarily, where it does not exist.

The correspondence between absolutely additive functions of plurisegments, $F(p)$, and absolutely additive functions of point sets measurable (B), $f(e)$, is (1, 1) if the discontinuities of the former are regular.† In order to show that the decomposition of an absolutely additive function, $f(e)$, of sets measurable (B) given by Theorem I holds for *any* $f(e)$ we shall assume that the discontinuities of $F(p)$ are of the first kind. In fact, we have, by the theorem referred to in this case, one and only one $F(p)$ corresponding to a given $f(e)$ and conversely. We may, therefore, apply Theorem I and obtain the decomposition of $f(e)$. We thus have the

THEOREM. *Every absolutely additive non-negative function, $f(e)$, of sets measurable* (B) *may be written as*

$$f(e)=S_e f+\int_e Df\,de\,.$$

The symbols have the same meaning as in Theorem I.

By means of Theorem H (§ 4) we have the following

COROLLARY. *The decomposition of an absolutely additive non-negativ function, $f(e)$, of sets measurable* (B) *into a function of sets measurable* (B) *whose derivative* $=0$ *almost everywhere in* Δ *and an absolutely continuous function of sets measurable* (B) *is unique.*

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* de la Vallée Poussin, *Intégrales de Lebesgue*, p. 73.

† G. C. Evans, The Rice Institute Pamphlet, vol. 7 (Oct., 1920). Nomenclature in the pamphlet is not the usual one. The discontinuities should be called regular instead of the first kind.

With the aid of the preceding corollary we see that $S_e f = f(eE)$, where $f(eE)$ is the function of singularities defined by de la Vallée Poussin.*

THEOREM. *Given an absolutely additive non-negative function $f(e)$ of normal sets† e : then*

$$f(e) = S_e f + \int_e Df\, de$$

for any normal measurable set in Δ.

In fact, $f(e)$, $S_e f$, $a(e)$ are completely defined by an extension of the definition of the symbols, such a definition being unique,‡ by the values which these functions have on sets measurable (B).§ Denote by $f'(e)$, $S'_e f$, $\int_e Df'de$, respectively, the functions which define $f(e)$, $S_e f$, $a(e)$. Then

$$f'(e) - S'_e f - \int_e Df'\, de = f(e) - S_e f - a(e) = 0$$

on every set measurable (B) and, therefore, on every normal measurable set. Since $a(e)$ is absolutely continuous as a function of normal measurable sets and since $DS_e f = 0$ almost everywhere in Δ it follows that

$$f(e) = S_e f + \int_e Df\, de\ .$$

6. Arbitrary additive functions of plurisegments

Now let $F(p)$ be any absolutely additive function of plurisegments, not necessarily positive. Then by Theorem B (§1) we can write

$$F(p) = U(p) - N(p)$$

where $U(p)$ and $N(p)$ are non-negative and of the same nature as $F(p)$. Therefore we can write for any $F(p)$

$$F(p) = f_2(p) - f_1(p) + \sum_i k_i^{(2)} \varphi_i^{(2)}(p) - \sum k_i^{(1)} \varphi_i^{(1)}(p) + \int_e [DF_2 - DF_1] d\sigma$$

where f_2, $\varphi_i^{(2)}$, F_2 refer to $U(p)$ and f_1, $\varphi_i^{(1)}$, F_1 refer to $N(p)$. In other words

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* de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 97.

† de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 83 ff.

‡ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 88.

§ de la Vallée Poussin, *Intégrales de Lebesgue*, 1916, p. 83 ff.

$$F(p)=f(p)+\sum_i k_i\varphi_i(p)+\int_p DF\,d\sigma\ ,$$

where the k_i are constants such that $\sum_i|k_i|$ is convergent. In fact DF exists wherever DF_1 and DF_2 both exist. The corresponding result is evidently true for any absolutely additive function of normal sets, i. e., $f(e)=S_e f+\int_e Df\,de$, $f(e)$ not necessarily positive.

Consider a plurisegment p in Δ and a set of positive numbers $\sigma_1>\sigma_2>\cdots$ such that $\lim_{n\to\infty}\sigma_n=0$. Now let ρ_n and λ_n be the upper bound and lower bound, respectively, of $F(p)$ for all finite plurisegments of measure $\leqq\sigma_n$ contained in p. Let $\lim_{n\to\infty}\rho_n=\rho$ and $\lim_{n\to\infty}\lambda_n=\lambda$. These limits obviously exist; then we see that ρ is the discard in p of the positive variation $U(p)$ of $F(p)$ and λ is the discard in p of the negative variation $N(p)$ of $F(p)$.

A slight examination of the preceding analysis shows that the results of this paper are valid for any finite number of dimensions.

Note: Given $F(p)$, an absolutely additive, non-negative function of plurisegments in Δ. Then assuming $f(e)=f(eE)+\int_e Df\,de$, where $f(e)$ is the function of point sets measurable (B) in Δ, we can show that $DS_pF=0$ almost everywhere in Δ.

It is not difficult to show that $f(\omega E)\geqq S_\omega F$ for any rectangle ω in Δ. Now, by hypothesis, $Df=0$ almost everywhere in Δ, and, therefore, $DSF=0$ almost everywhere in Δ. It would then follow that

$$F(p)=S_pF+\int_p DF\,d\sigma\ .$$

It is to be noted that this is an alternative proof. Preference is given to the other method, because it enables us to obtain the decomposition of a function of point sets directly from that of a function of plurisegments.

7. Functions of plurisegments in curved space

We shall confine the discussion to a somewhat restricted class of surfaces. To consider the typical case we take a surface, in three-dimensional space, defined by the equations

$$x_i=x_i(u,v) \qquad (i=1,2,3)\ .$$

We assume x_i to be a continuous function of u, v and to have continuous first partial derivatives with respect to u and v.

In order to specify points on the surface we trace upon the surface two families of curves, the locus of points for which one or the other of the variables u, v remains constant. Call these $u=\text{const.}$, $v=\text{const.}$ curves, respectively. We assume that each $u=\text{const.}$ curve intersects every $v=\text{const.}$ curve, and conversely.

DEFINITION. A singular point of the coördinate system (u, v) is a point P on the surface through which pass all of the $v=\text{const.}$ curves, but no $u=\text{const.}$ curve other than the curve which is the point P itself. We may, without sacrifice of clearness, call such a point a pole. It is assumed that there are none, one, or two such points on the surface. Through a point of the surface not a singular point passes one and only one $u=\text{const.}$ curve and one and only one $v=\text{const.}$ curve. A pole P_0 is uniquely determined by $u=u_0$.

Consider a point on the surface which is not a pole. Through this point, by hypothesis, passes $u=c_1$, $v=c_2$. From this point we measure an arc length s along $u=c_1$ and assume the coördinate system to be such that if $v=c_2'$ passes through a point on $u=c_1$ corresponding to a value of s_2' of arc length and $v=c_2''$ passes through a point corresponding to a value s_2'' of arc length we shall have $c_2''>c_2'$ if $s_2''>s_2'$. A similar situation is assumed if we go along a $v=\text{const.}$ curve. We take for positive sense along a $v=\text{const.}$ curve the direction in which the parameter s increases, and likewise upon a $u=\text{const.}$ curve the direction in which s increases. It is seen from the assumption that this definition is not ambiguous.

If either one or both of the systems $u=c_1$, $v=c_2$ of curves are closed, then $u=c_1\pm n\omega_1$, $v=c_2\pm m\omega_2$, n and m positive integers, will mean the same curve if ω_1 or ω_2 is the period of u or v.

Let $u=u_0$ determine the pole P_0. We then take the curves $u=u_0+\eta$ and assume that for all $0<\eta<\epsilon$, $\epsilon>0$ and sufficiently small, the curves given by $u=u_0+\eta$ are closed. We have tacitly assumed that η is to be measured along the positive direction of a $v=\text{const.}$ curve from u_0. Any point on the surface with coördinates (u, v) and such that $u_0\leqq u\leqq u_0+\eta$ is said to be interior to the curve $u=u_0+\eta$. The configuration composed of the interior points of $u=u_0+\eta$ together with its boundary points is said to be a polar cap. We shall denote this configuration by the symbol $C_\eta(P_0)$.

DEFINITION. Through a point (c_1, c_1'), not a pole, of the surface draw two curves $u=c_1$, $v=c_1'$. Take a point on $u=c_1$ not (c_1, c_1'), and draw through it the curve $v=c_2'$. Take a point on $v=c_1'$ not (c_1, c_1'), and draw through it the curve $u=c_2$. The curves $u=c_2$, $v=c_2'$ intersect by hypothesis. The set of points E_I on the surface having coördinates (u, v) which satisfy the relations

$$c_1<u<c_2, \quad c_1'<v<c_2'$$

are said to be interior to the quadrilateral determined by $u=c_1$, $u=c_2$, $v=c_1'$, $v=c_2'$. In order to avoid ambiguity we assume that $c_2-c_1<\omega_1$, $c_2'-c_1'<\omega_2$. Let E_B be the set of boundary points of E_I. It is assumed that $E_s=E_I+E_B$ contains no pole.

DEFINITION. The configuration determined by the set E_s shall be called a segment S_i. The points of S_i which have the coördinates (c_1, c_1'), (c_1, c_2'), (c_2, c_1'), (c_2, c_2') are called the vertices of S_i. If the vertices of an S_i have the coördinates $(u+h, v+h)$, $(u-h, v-h)$, $(u-h, v+h)$, $(u+h, v-h)$ where u, v are the coördinates of a point P interior to S_i and $h>0$, then the segment S_i is said to be a regular segment relative to P. We denote such a segment by the symbol $S(P, h)$.

We write

$$E=\sum_i\left(\frac{\partial x_i}{\partial u}\right)^2, \qquad F=\sum_i\frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}, \qquad G=\sum_i\left(\frac{\partial x_i}{\partial v}\right)^2.$$

The measure of any segment S_i or polar cap $C_\eta(P_0)$ is then

$$\iint_{S_i}\sqrt{EG-F^2}\,dudv=m(S_i)\ .$$

The expression $EG-F^2$ is assumed to be positive for any point of the surface not a pole. It is easily seen that $H=\sqrt{EG-F^2}=0$ at a pole.

Measure of point sets on a surface. It is clear that the descriptive properties of point sets on a two-dimensional surface are the same as those of point sets in a three-dimensional euclidean space. As for the theory of measure we may define the exterior measure of a bounded set E, i. e., E is contained in a finite volume of three-dimensional space, as the lower bound of the measure of all open sets which contain E. Even in this case this quantity may be infinite. An open set can be shown to be composed of a denumerable infinity of non-overlapping segments, possibly having boundary points in common, if we extend the definition of segments to include the triangular segments which have a pole as vertex. We take as the measure of a segment $m(S_i)$ defined above. The interior measure of E is defined as the upper bound of the measure of all closed sets contained in E. If the exterior is equal to the interior measure then E is said to be measurable.

THEOREM. *Let E be an arbitrary bounded set of finite but not of zero exterior measure and let O be an open set containing E. To each point P of E (not a pole) is assigned a set of regular segments*

$$S_1(P, h_1)\ , \quad S_2(P, h_2)\ , \ \cdots$$

such that $h_1 > h_2 > h_3 > \cdots$ *where* $\lim_{i\to\infty} h_i = 0$. *To a pole* P *we assign a set of polar caps*

$$C\eta_1(P_0)\ , \quad C\eta_2(P_0)\ , \ \cdots$$

such that $\lim_{i\to\infty} m\,[C\eta_i(P_0)] = 0$.

If we denote by $\bar{E}$ the set E less the points of E which are interior or on the boundary of the polar caps of arbitrarily small measure, we see that we can assume that the oscillation of H on

$$S_1(P, 3h_1) < \epsilon(\text{minimum } H \text{ on } \bar{E}),\ 1 > \epsilon > 0$$

uniformly with respect to P, if only h_1 is sufficiently small, say $h_1 < \delta$ independently of P.

Then if θ is arbitrarily near unity we can find a finite number of points of E

$$P_1\ , P_2\ , \ \cdots\ , P_m$$

and just as many integers $k_1, k_2, \cdots, k_m$, so that if

$$S_i' = S_{k_i}(P_i h_{k_i}) \qquad (i = 1, 2, \cdots, m)$$

we shall have

$$S_i' < O\ , \qquad S_i' \cdot S_j' = 0\ , \qquad i \neq j\ .$$

Furthermore, if

$$S = \sum_1^m S_i'$$

then

$$m_e(ES) > \theta m_e E$$

and accordingly

$$m_e(E - E \cdot S) < (1-\theta) m_e E.$$

The above theorem is an extension of Vitali's Lemma to surfaces and the proof of it does not present any new difficulties.

DEFINITION. The collection of a finite or denumerable infinity of segments two by two distinct but having, possibly, boundary points in common, together with none, one, or two polar caps shall be called a plurisegment p.

The portion of the surface considered in this theory which we shall designate as the fundamental region is understood to be contained in a bounded portion of euclidean three-space and to be of finite measure.

The definitions used for functions of plurisegments on a surface are nearly the same as for the plane case. In fact, except for point values and the derivative, it is merely necessary to understand surface for plane.

DEFINITION. A set of plurisegments $\{p_i\}$ is said to form a regular family relative to P if

$$\frac{m(P_i)}{mS(P,h_i)} > \alpha\ ,\ \frac{m(P_i)}{mC_\eta(P)} > \alpha$$

according as P is not or is a pole, where $\alpha > 0$ is independent of i and $S(P, h_i)$ or $C_\eta(P)$ is the smallest regular segment or polar cap relative to P and containing p_i.

DEFINITION. The derivative of $F(p)$ at a point P is defined as $\lim_{m(p_i)\to 0} F(p_i)/m(p_i)$, if it exists, where $\{p_i\}$ form a regular family relative to P.

DEFINITION. The upper symmetric derivative of $F(p)$ at P is defined as

$$\overline{\lim_{h\to 0}} \frac{F[S(P,h)]}{m[S(P,h)]} \text{ or } \overline{\lim_{\eta\to 0}} \frac{F[C(P)]}{m[C_\eta(P)]}$$

according as P is not or is a pole.

For point values we use the same definition as in the plane case except that now it is not necessary to distribute the point value at a pole among various segments, since such a point is strictly interior to a polar cap or is a boundary point of a plurisegment.

By an examination of the theory for absolutely additive functions of plurisegments in the plane it is seen that

$$F(p) = \sum k_i^{(2)} \varphi_i^{(2)}(p) - \sum k_i^{(1)} \varphi_i^{(1)}(p) + f_2(p) - f_1(p) + \int_p D[F_2 - F_1]d\sigma$$

where the meaning of the symbols is clear after what has been said.

In conclusion the author desires to express his obligations to Professors Evans and Bray for criticisms and suggestions.

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SETS OF POSTULATES FOR THE LOGIC OF PROPOSITIONS*

BY

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1. **Introduction.** Boole† created the mathematical sciences now known as the logic of classes and the logic of propositions.‡ Peirce§ and Schröder‖ have perfected these sciences. The logic of classes now rests on rigorous, postulational foundations. Various sets of postulates, with consistency, independence, and sufficiency proved, have been worked out for that logic.¶ But no rigorous postulate sets exist for the logic of propositions.** The object of this paper is to supply such sets for this logic.

My starting point is Schröder's analysis of the logic.†† The propositional logic is the body of facts concerning the truth or falsity of propositions $a, b, c, \cdots$ and their combinations. Every proposition a is either true or else it is false. If true, we write $a=1$; if false, we write $a=0$. Every fact in this logic, symbolically expressed, is also a fact in the logic of classes consisting of the elements 0, 1; and conversely. With Schröder, therefore, I define the logic of propositions to be a two-element logic of classes. It is the algebra of truth-values 0, 1.‡‡

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† George Boole, *An Investigation of the Laws of Thought*, London, 1854.

‡ Boole develops these logics as theories of "primary" and "secondary" propositions. Boole's theories, as improved by Peirce and Schröder, may be called, after Sheffer, *boolean algebras*.

§ C. S. Peirce. For a list of Peirce's writings see M. R. Cohen, Journal of Philosophy, Psychology, and Scientific Methods, vol. 13 (1916), p. 733.

‖ E. Schröder. See his *Vorlesungen über die Algebra der Logik*, Leipzig, vol. 1, 1890; vol. 2, part 1, 1891; vol. 2, part 2, edited by E. Müller, 1905. Vol. 2, part 1, will be referred to in the later footnotes simply as *Vorlesungen*.

¶ See, e. g., E. V. Huntington, these Transactions, vol. 5 (1904), p. 288; H. M. Sheffer, ibid., vol. 14 (1913), p. 481; B. A. Bernstein, ibid., vol. 17 (1916), p. 50.

** Boole states inaccurately (*Laws of Thought*, p. 159) that the formal laws underlying his theory of primary propositions are identical with those underlying his theory of secondary propositions. Schröder is mistaken in thinking (*Vorlesungen*, pp. 51–52) that by adding to any postulate set for the class logic a postulate to the effect that the logic consists of two elements, a set of *independent* postulates will thereby be obtained for the propositional logic. Whitehead and Russell's set of primitives for their "theory of deduction" (*Principia Mathematica*, vol. 1, pp. 94–101) is imperfect. The authors fail, among other things, to give a satisfactory discussion of the independence of the primitive propositions (see a later footnote).

†† See *Vorlesungen*, §§21–31.

‡‡ Schröder and other mathematical logicians do not distinguish between the symbols $<, +, \times$ that *belong* to the propositional logic (either as primitive ideas or as ideas defined by means of the primitive ideas) and the notions "implies," "or," "and" respectively that are *outside* the logic. Thus if a, b, c are elements of our logic and "$<$" the relation of "implication" belonging to the logic, then we have the proposition

(i) If $a<b$ and $b<c$, then $a<c$.

Four sets of postulates are obtained. If we denote by K a class of elements and by $+$, $\times$, $|$ the operations of logical addition, multiplication, rejection* respectively, the undefined notions in the sets are

$$K, +, \times; \quad K, |; \quad K, +; \quad K.$$

(For convenience $a \times b$ will be written $a \cdot b$ or ab.) The postulates in these sets are in number respectively five, four, three, one. As will be seen, the first two sets are more than a mere selection from corresponding postulate sets for the general logic of classes. The fourth set is quite novel. It consists of a single postulate involving a single undefined notion, the notion of *class*.

For each set, the complete existential theory† of the postulates is established, Tables I, II, III exhibiting the existential theories of the first three sets. In these tables the signs $+$, $\times$ at the head of the columns are enclosed in circles, in order to distinguish them from the arithmetic $+$, $\times$ employed in the concrete systems found in the body of the tables. In the

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Or, writing "implies" for "$<$", we have

(ii) If "a implies b" and "b implies c," then "a implies c."

But Schröder (*Vorlesungen*, p. 28) would write (ii) in the form

(iii) $$[(a<b)\times(b<c)]<(a<c),$$

where the "if . . . then" and "and" of (ii), which are outside the logic, are replaced by the symbols "$<$" and "$\times$" of the logic. It is this confusion between the symbols of the logic of propositions and notions outside the logic that leads Schröder (*Vorlesungen*, pp. 45–48) to go through a lengthy explanation of why the duality principle that exists in the logic of classes breaks down in the logic of propositions when, as a matter of fact, it does not break down!

It is the failure to differentiate properly between the symbols of the propositional logic and notions outside the logic that causes Whitehead and Russell in their *Principia*, vol. 1, to hold (p. 94) that the theory of propositions is more fundamental than the theory of classes and (p. 95, footnote) that the recognized methods of proving the independence of a set of postulates are inapplicable to the primitive propositions of the logic of propositions. If P is the set of postulates of any mathematical science Σ, and Q the body of propositions deducible from P, then, as Russell has well said somewhere, the science Σ is simply the proposition "P implies Q." Now the relation "implies" in this proposition is *outside* Σ, even if Σ is the logic of propositions in which a relation "$<$" of "implication" is involved. Indeed, the notions "proposition," "negation," etc. and the syllogism, the law of contradiction, etc. are all used in unsymbolized forms in deducing these very laws symbolically from an appropriate set of postulates for the logic of propositions. The unsymbolized forms of these notions and laws belong to a general logic, a sub-logic, underlying the logic that is receiving symbolic treatment. And there is no circularity in all this. On the contrary, one must recognize a sub-logic as underlying any symbolic logic, if he is to avoid circularity. The recognition of this sub-logic is essentially the observance of Russell's own principle, used by him so effectively in the solution of paradoxes, the principle that no collection may contain the collection itself as a member. And so, proper differentiation between the symbols and laws of a propositional logic and the corresponding ideas and laws of its sub-logic would lead to the exclusion from the set of primitive ideas and primitive propositions for the propositional logic of the *Principia* notions such as "assertion" and propositions such as "Anything implied by a true elementary proposition is true." The resulting set of primitives would not be in the least more fundamental for logic than a set of primitives for the logic of classes, and the usual methods of proving independence would be applicable to it.

* For the concrete meaning of this operation see H. M. Sheffer, loc. cit.

† See E. H. Moore, *Introduction to a Form of General Analysis*, New Haven Colloquium, Yale University Press, p. 82.

concrete systems, an operation followed by "mod 3" (or "mod 2") denotes the least positive integer obtained from the operation by dropping multiples of 3 (or 2).

2. **Set I. Postulates in terms of addition and multiplication.** The following postulate set for the logic of propositions is modelled after Huntington's first set* of ten postulates for the general logic of classes. It leaves undefined class K and operations $+$, $\times$, and it exhibits the law of duality in the logic. Note that postulates 3, 4 are not found in Huntington's set.

1. There is a K-element 0 such that $0+a=a$ for every K-element a.
2. There is a K-element 1 such that $1 \cdot a=a$ for every K-element a.
3. If an element 0 of postulate 1 exists, then $0 \cdot a=0$ for every K-element a.
4. If an element 1 of postulate 2 exists, then $1+a=1$ for every K-element a.
5. K consists of two elements.

3. **Sufficiency of Set I.** Propositions 6–14 below, derived from Set I, together with postulates 1, 2, 5, give Huntington's first set of postulates for the general logic of classes, and hence prove the sufficiency of Set I for the logic of propositions.

6. $0 \neq 1$.

For if $0=1$, take (by 5) a K-element a not 0; then $a=1 \cdot a=0 \cdot a=0$, by 2, hyp., 3.

7. 0 and 1 are unique.

For K has no elements other than 0, 1, by 5, 6. And $1+0=1 \neq 0$, by 4, 6; $0 \cdot 1=0 \neq 1$, by 3, 6.

8. $a+b$ is a K-element whenever a, b are K-elements.

For $0+b=b$, $1+b=1$, by 1, 4.

9. ab is a K-element whenever a, b are K-elements.

For $0 \cdot b=0$, $1 \cdot b=b$, by 3, 2.

10. $a+b=b+a$.

For $0+1=1=1+0$, by 1, 4.

11. $ab=ba$.

For $0 \cdot 1=0=1 \cdot 0$, by 3, 2.

12. $a+bc=(a+b)(a+c)$.

For $0+bc=bc=(0+b)(0+c)$, by 1, 1; $1+bc=1=1 \cdot 1=(1+b)(1+c)$, by 4, 2, 4.

13. $a(b+c)=ab+ac$.

For $0(b+c)=0=0+0=0 \cdot b+0 \cdot c$, by 3, 1, 3; $1(b+c)=b+c=1b+1c$, by 2, 2.

* E. V. Huntington, loc. cit.

14. For every K-element a there is a K-element a' such that $a+a'=1$, $aa'=0$.

For $0+1=1$, $0\cdot 1=0$, by 1, 3; $1+0=1$, $1\cdot 0=0$, by 4, 2.

Therefore, $0'=1$, $1'=0$.

4. **Complete existential theory of Set I.** Table I below exhibits the complete existential theory of the postulates of Set I. System 32 proves the consistency of the postulates and systems 16, 24, 28, 30, 31 prove their (ordinary) independence. The non-existence (indicated by blanks) of systems having the characters $(-\pm-\pm\pm)$, $(\pm-\pm-\pm)$ is in accordance with the evident facts that postulate 3 cannot be contradicted if postulate 1 is denied, and that postulate 4 cannot be contradicted if postulate 2 is denied.

Table I

Existential Theory of Set I

No.	Character	K	$a \oplus b$	$a \odot b$
1	(− − − − −)	—	—	—
2	(− − − − +)	—	—	—
3	(− − − + −)	—	—	—
4	(− − − + +)	—	—	—
5	(− − + − −)	—	—	—
6	(− − + − +)	—	—	—
7	(− − + + −)	0	0/0	0/0
8	(− − + + +)	0, 1	0/0	0/0
9	(− + − − −)	—	—	—
10	(− + − − +)	—	—	—
11	(− + − + −)	—	—	—
12	(− + − + +)	—	—	—
13	(− + + − −)	0	0/0	0
14	(− + + − +)	0, 1	0	$a\,b$
15	(− + + + −)	0, 1, 2	1	$a\,b \bmod 3$
16	(− + + + +)	0, 1	1	$a\,b$
17	(+ − − − −)	—	—	—
18	(+ − − − +)	—	—	—
19	(+ − − + −)	0	0	0/0
20	(+ − − + +)	0, 1	$a\,b$	0
21	(+ − + − −)	—	—	—
22	(+ − + − +)	—	—	—
23	(+ − + + −)	0, 1, 2	$a\,b \bmod 3$	1
24	(+ − + + +)	0, 1	$a\,b$	1
25	(+ + − − −)	0, 1, 2	$a+b \bmod 3$	$a+b+2 \bmod 3$
26	(+ + − − +)	0, 1	$ab+b \bmod 2$	$ab+a+1 \bmod 2$
27	(+ + − + −)	0, 1, 2	$2a^2b+a+b \bmod 3$	$a+b+2 \bmod 3$
28	(+ + − + +)	0, 1	$ab+a+b \bmod 2$	$a+b+1 \bmod 2$
29	(+ + + − −)	0, 1, 2	$a+b+2 \bmod 3$	$2a^2b+a+b \bmod 3$
30	(+ + + − +)	0, 1	$a+b+1 \bmod 2$	$ab+a+b \bmod 2$
31	(+ + + + −)	0	0	0
32	(+ + + + +)	0, 1	$ab+a+b \bmod 2$	ab

5. **Set II. Postulates in terms of rejection.** A set of independent postulates sufficient for the logic of propositions, expressed in terms of *class K* and Sheffer's operation of *rejection* "|", is the set *A–D* below. Note how much simpler this set is than the corresponding sets given by Sheffer* and by the writer† for boolean algebras in general.

A. $a|b$ is a unique K-element whenever a, b are K-elements.

DEFINITION. $a'=a|a$.

B. $a'\neq a$ whenever a, $a|a$ are K-elements.

C. $a|b=b|a$ whenever a, b, $a|b$, $b|a$ are K-elements, and $a\neq b$.

D. K consists of two elements.

6. **Sufficiency of Set II.** Theorems *E–K* following, derived from Set II, prove the sufficiency of this set for our logic of propositions, since propositions A, D, J, K give the set of postulates previously proved by the writer‡ sufficient for boolean algebras. In the proofs of the theorems reference to postulate A will generally be omitted.

E. $(a')'=a$.

By A, D, B.

F. $a|a'=a'|a=b|b'$.

For $a|a'=a'|a$, by B, C; if $b\neq a$, then $b=a'$, by D, B, hence $b|b'=a'|(a')'=a'|a$, by E.

DEFINITIONS. $0=a|a'$; $1=0'$.

G. $1'=0$.

By E.

H. $a|0=0|a=a'$.

For $0|0=0'$, by def.; $1|0=1|1'=0=1'$, by G, def., G.

I. $a|1=1|a=0$.

For $0|1=1'=0$, by H, G; $1|1=1'=0$, by def., G.

J. $(b|a)|(b'|a)=a$.

For $(0|a)|(0'|a)=a'|(1|a)=a'|0=(a')'=a$, by H, def., I, H, E; $(1|a)|(1'|a)=0|(0|a)=0|a'=(a')'=a$, by I, G, H, H, E.

K. $a'|(b'|c)=[(b|a')|(c'|a')]'$.

For $0'|(b'|c)=1|(b'|c)=0$, by def., I; $[(b|0')|(c'|0')]'=[(b|1)|(c'|1)]'=(0|0)'=(0')'=0$, by def., I, def., E. Also $1'|(b'|c)=0|(b'|c)=(b'|c)'$, by G, H; $[(b|1')|(c'|1')]'=[(b|0)|(c'|0)]'=[b'|(c')']'=(b'|c)'$, by G, H, E.

7. **Complete existential theory of Set II.** Table II following establishes the complete existential theory of Set II. The postulates of this set are

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* Loc. cit.

† Loc. cit.

‡ Loc. cit.

seen to be *completely independent*,* system 16 proving the consistency of the postulates and systems 8, 12, 14, 15 proving their (ordinary) independence.

Table II

Existential Theory of Set II

No.	Character	K	$a \mid b$	No.	Character	K	$a \mid b$
1	$(----)$	0,1,2	$b+0/b$	9	$(+---)$	0,1,2	b
2	$(---+)$	0,1	$b+0/(ab+a+b) \bmod 2$	10	$(+--+)$	0,1	b
3	$(--+-)$	0,1,2	$1+0/b$	11	$(+-+-)$	0	0
4	$(--++)$	0,1	$1+0/(ab+a+b) \bmod 2$	12	$(+-++)$	0,1	$ab+a+b \bmod 2$
5	$(-+--)$	0,1,2	$b+2+0/b \bmod 3$	13	$(++--)$	0,1,2	$b+1 \bmod 3$
6	$(-+-+)$	0,1	$a+1+0/(ab+a+b) \bmod 2$	14	$(++-+)$	0,1	$a+1 \bmod 2$
7	$(-++-)$	0	0/0	15	$(+++-)$	null	—
8	$(-+++)$	0,1	$0+0/(ab+a+b) \bmod 2$	16	$(++++)$	0,1	$ab+a+b+1 \bmod 2$

8. **Set III. Postulates in terms of addition.** Corresponding to Huntington's set of nine postulates† for the general logic of classes expressed in terms of *class* K and operation of *addition* we may use for the logic of propositions the three postulates α-γ following.

α. There is a K-element 0 such that $0+a=a$ for every K-element a.

β. There is a K-element 1 such that $1+a=1$ for every K-element a.

γ. K consists of two elements.

9. **Sufficiency of Set III.** We have, first,

δ. $1 \neq 0$.

For otherwise, if x be any element, $x=0+x=1+x=1$, by α, β. But this is contrary to γ.

DEFINITIONS. $0'=1$; $1'=0$; $ab=(a'+b')'$.

The sufficiency of Set III for the logic of propositions then follows from Theorems η, ζ, θ below, since ζ, θ are postulates 2, 3 respectively of Set I.

η. $(a')'=a$.

For $(0')'=1'=0$ by def., def.; $(1')'=0'=1$, by def., def.

ζ. $1a=a$.

For $1a=(1'+a')'=(0+a')'=(a')'=a$, by def., def., α, η.

θ. $0 \cdot a=0$.

For $0 \cdot a=(0'+a')'=(1+a')'=1'=0$, by def., def., β, def.

10. **Complete existential theory of Set III.** The complete existential theory of Set III is given by Table III following. The set is seen to be completely independent.

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* E. H. Moore, loc. cit.

† E. V. Huntington, loc. cit.

Table III

Existential Theory of Set III

No.	Character	K	$a \oplus b$	No.	Character	K	$a \oplus b$
1	$---$	0	0/0	5	$+--$	0,1,2	b
2	$--+$	0,1	$b+1 \bmod 2$	6	$+-+$	0,1	b
3	$-+-$	0,1,2	1	7	$++-$	0	0
4	$-++$	0,1	1	8	$+++$	0,1	$ab+a+b \bmod 2$

11. **Set IV. A one-postulate set.** The following set consists of the single postulate P, in which the notion of *class K* is the only notion left undefined. The operations having the properties of logical addition and logical multiplication are both defined by means of P.

POSTULATE P. K consists of two elements.

DEFINITIONS.

1. Denote the two K-elements by 0, 1.
2. $a+b$: $0+0=0$, $0+1=1$, $1+0=1$, $1+1=1$.
3. ab: $0 \cdot 0=0$, $0 \cdot 1=0$, $1 \cdot 0=0$, $1 \cdot 1=1$.

Postulates 1–5 of Set I then follow, so that postulate P is sufficient for our logic.

12. **A new principle in postulate theory.** The procedure employed in § 11 makes evident the following new principle in postulate theory: *If a system is finite, a single postulate, expressed in terms of the single undefined notion of class, is sufficient for the system.*

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MULTIPLE-SHEETED SPACES AND MANIFOLDS OF STATES OF MOTION*

BY
HAROLD HOTELLING

Every closed orientable manifold can be represented topologically by a generalized Riemann surface consisting of a number of spaces, each with a single point at infinity, which are considered superimposed and which are joined with each other along certain branch cuts of dimensionality one less than that of the manifold. This has been proved by Professor J. W. Alexander,† who also pointed out that the branch cuts' boundaries may be taken as non-singular non-intersecting manifolds.

But the straightforward reduction of a manifold otherwise defined to such a form is likely to be very tedious and to yield a space having an unnecessarily large number of sheets. Consider for example the three-dimensional manifolds of states of motion used by Poincaré and Birkhoff in connection with dynamical problems having two degrees of freedom and studied topologically by the present writer in a paper‡ referred to hereinafter as A. In following the proof by Alexander of the existence theorem mentioned, the most obvious way is to represent the manifold by a polyhedron as in A, and to divide each face into triangles and the manifold into tetrahedra having a common vertex and these triangles as bases. The vertices of each tetrahedron must all be distinct points. The manifold of states of motion on a surface of genus p is in this way divided into some $24p-4$ tetrahedra. After fitting together all these cells to form a multiple-sheeted space there will remain the not inconsiderable task of so altering the branch system as to get rid of singularities. An attempt to carry through these operations is enough to show the need of shorter methods in applications.

With the aid of the theorems of §§ 1 to 6, multiple-sheeted spaces with non-singular branch systems can easily be set up for many three-dimensional manifolds, including manifolds of states of motion, orientable product manifolds, and all manifolds of genus unity. The application to manifolds

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† Bulletin of the American Mathematical Society, vol. 26 (1920), pp. 370–372.

‡ *Three-dimensional manifolds of states of motion*, these Transactions, vol. 26 (1925), pp. 329–344.

of genus unity is made in §§ 5 and 6, to manifolds of states of motion in §§7 to 10, and to product manifolds in § 11.

The term "covering space," which has been suggested by Kérékjárto, will be used to denote these generalized Riemann surfaces. It seems a better name than "Riemann space," which is in use in a very different sense.

A covering space representing a manifold is never unique, since the number of sheets can be increased in various ways without altering the topology of the manifold. How far, on the other hand, the number of sheets can be reduced is one of the unanswered questions of analysis situs, except that in the two-dimensional case it is known that the number can always be reduced to two. Whether the covering spaces here found to represent various manifolds are in this sense the simplest possible cannot, therefore, now be known. But it is difficult to see how any further simplification is possible.

1. We shall use special coördinates ρ, θ, φ derived from ordinary cylindrical coördinates r, θ, z, by the equations

$$r-a=\rho\cos\varphi,$$
$$z=\rho\sin\varphi,$$

where a is a positive number.

The locus for which $\rho=b$, where b is a constant less than a, is a tore. A closed sensed curve deformable on this tore into a sensed curve for which θ is constant and φ increases through 2π as the curve is traversed in the positive sense will hereafter be referred to as an α-curve. It bounds a 2-cell in the interior of the tore. Any closed sensed curve deformable on the tore into a sensed curve for which φ is constant and θ increases through 2π as the curve is traversed in the positive sense will be called a β-curve. A β-curve evidently bounds a 2-cell on the exterior of the tore. We shall also call a curve on any surface of genus 1 which bounds a 2-cell on its interior an α-curve, and one which bounds a 2-cell on its exterior, a β-curve. A closed curve deformable into one which traverses an α-curve m times and a β-curve n times—operations which by a fundamental property of the tore are commutative—will be called $\alpha^m\beta^n$.

Consider now a strip generated by the motion of a straight segment which at each stage of the motion lies in a plane $\theta=$constant, with one end on the curve $\varphi=n\theta$, $\rho=c$, and the other on the curve $\varphi=n\theta+\pi$, $\rho=c$, where $c\leqq b$. Such a strip will be said to have an n-fold twist. If n is a positive or negative integer or zero the boundaries are distinct curves which are said to have linkage with each other of order n. The same terminology will be used for any figure with which this is isotopic. If n is an odd multiple of $\frac{1}{2}$, the strip

has only one boundary and is one-sided. We shall consider only non-singular strips, and consequently no other values of n will appear.

Let two sheets cover a space of inversion and join each other along a strip having an $(n/2)$-fold twist. Let a tore which surrounds the strip in one sheet be deformed continuously until it lies on the strip, covering it doubly; let it then pass into the other sheet and finally cover the initial position. This deformation determines a transformation Δ_n into itself of a tore in the fundamental space which is covered by the initial and final positions of the moving tore.

This transformation Δ_n is such that an α-curve goes into a curve α^{-1} and a β-curve goes into a curve $\alpha^n\beta$; a statement which we may write

$$\alpha' = \alpha^{-1},$$
$$\beta' = \alpha^n\beta;$$

or in matrix notation

$$\Delta_n = \begin{Vmatrix} -1 & 0 \\ n & 1 \end{Vmatrix}.$$

To prove this we observe that the deformation may be performed in such a way that each point of the tore in the fundamental space goes into a point which lies in the same half-plane $\theta =$ constant, and is the image of its transform under a reflection on the segment in which this half-plane meets the strip. The point (ρ, θ, φ) then goes into the point (ρ, θ, φ'), where $\varphi' + \varphi = n\theta$. Consequently a β-curve for which φ is constant goes into a curve for which φ' varies through an angle $2n\pi$ as θ varies through 2π, and is therefore a curve of type $\alpha^n\beta$. An α-curve obviously goes into its inverse.

2. The region between two coaxial tores of which one is within the other is carried by an inversion about a point P of this region, followed by a deformation, into the region exterior to a pair of linking tores. For since each tore is accessible from P before the inversion, each must be accessible from the point at infinity afterwards. Also it is a property of inversions to carry linking circles into linking circles; so that, because certain circles of one tore link certain circles of the other before the inversion, the same must be true afterwards, so that the tores must link.

3. A consequence of this, to be used in §§ 7 and 8, is that the region between a tore and p others which are coaxial with and within it, but exterior to each other, is homeomorphic with the region exterior to $p+1$ tores of which one links all the rest but which are not otherwise linking.

4. Another consequence of § 2 is that a tore T may be deformed into a tore T' which it links, through a non-singular set of intermediate posi-

tions which fill the space exterior to the two tores. The transformation D of T into T' is such that an α-curve and a β-curve of T go respectively into a β-curve and an α-curve of T', provided we define appropriately the senses of these curves. We write

$$D = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

and note in passing that $D^{-1}=D$ and that $\Delta_n^{-1}=\Delta_n$.

5. A manifold of genus 1 is one whose Heegaard diagram* is a tore. The interior of this tore represents a part of the manifold; on its surface is traced a characteristic curve, which bounds in the other part.

If the characteristic curve is of type $\alpha\beta^n$ the manifold is homeomorphic with a two-sheeted covering space whose sheets join along a strip having an $(n/2)$-fold twist. For let the tore of the diagram link the strip, and let the region be deformed in such a manner that its surface undergoes first the transformation D and then Δ_n. Thus $\alpha\beta^n$ is transformed first into $\alpha^n\beta$ and then into β, which bounds in the remainder of the manifold, the exterior of the tore in the second sheet.

6. We shall now show that any manifold of genus 1 is represented by a sequence of sheets of which each joins each of its neighbors in the sequence along a twisted strip. In this covering space the sequence of sheets determines a sequence of strips of which each links in the simplest manner each of its neighbors in the sequence. A tore in the first sheet, linking the first strip, may by alternate applications of the deformations of § 1 and § 4 be deformed into a tore in the last sheet which links the last strip. In this process all the intermediate sheets are swept out. If the successive strips have twists of orders $q_1/2, q_2/2, \cdots, q_p/2$, the transformation of the initial position of the tore into the final position is $S=D\Delta_{q_p}D\Delta_{q_{p-1}}\cdots D\Delta_{q_1}D$.

Let the characteristic curve on the Heegaard diagram be $\alpha^m\beta^n$. Our proposition will be proved if we can find numbers $q_1, \cdots, q_r$, positive or negative integers or zero, such that $S(\alpha^m\beta^n)=\alpha$ or α^{-1}.

The transformation of one into the other of the two surfaces bounding regions each of which represents a part of the manifold may be written

$$\alpha' = \alpha^m\beta^n,$$
$$\beta' = \alpha^s\beta^t,$$

where, since the transformation is reciprocally one-to-one, $mt-ns=\pm 1$.

* Cf. *A*, § 2.

Hence m and n are prime to each other. Without loss of generality we shall assume m and n positive and $n>m$.

Now $D\Delta_{q_1}D(\alpha^m\beta^n)=\alpha^m\beta^{q_1m-n}$. If we let q_1 equal the integral quotient obtained in dividing n by m, the exponent of β becomes the negative of the remainder r_1 and is therefore numerically less than m. If we then apply the transformation $D\Delta_{q_2}$ to $\alpha^m\beta^{-r_1}$, we obtain $\alpha^{-r_1}\beta^{-m-r_1q_2}$. Here let q_2 be the negative of the quotient obtained in dividing m by r_1. The exponent of β in the last expression is the negative of the remainder, r_2, and so is less than r_1. Proceeding in this way we apply successively the transformations $D\Delta_{q_i}$, letting q_i take the value of the ith quotient obtained in the process of finding the greatest common divisor of m and n, multiplied by $(-1)^i$. If we denote the successive remainders in this process by r_1, r_2, etc., $\alpha^m\beta^n$ is transformed successively into

$$\alpha^m\beta^{-r_1},\quad \alpha^{-r_1}\beta^{-r_2},\quad \alpha^{-r_2}\beta^{r_3},\quad \alpha^{r_3}\beta^{r_4},\text{ etc.},$$

the exponents continually decreasing in absolute value. Since m and n are prime to each other, one of the letters α, β will eventually take 1 or -1 as its exponent; the other exponent will then become zero. If $\alpha^m\beta^n$ is in this way transformed into $\beta^{\pm1}$, a further transformation $D\Delta_0$ will carry it into $\alpha^{\pm1}$. Having determined the values of $q_1,\cdots,q_p$ which accomplish this we have proved the proposition.

Use has been made of the fact that α and β are commutative. The absence of this property for curves on surfaces of genus greater than one prevents an immediate generalization of the method.

Manifolds of states of motion

7. The motion of a particle on a surface at any instant is specified by three parameters, two giving the position on the surface and one the direction; the energy is assumed constant. These three parameters also specify a point in a manifold whose topological properties depend only on those of the portion of the characteristic surface on which the particle's total energy is positive. Covering spaces representing these manifolds have been found in *A* for the cases in which the connectivity of the surface does not exceed that of the tore, and also for all cases in which there are boundaries. With the help of §§ 1 to 4 we now find covering spaces representing manifolds of states of motion on the remaining surfaces.

Divide an orientable surface of genus p into two parts by $p+1$ non-intersecting non-homologous curves and map each part conformally on a plane region within a circle and outside of p smaller circles. The states of motion on one of these parts are represented by the points of a region R

between a tore and p smaller tores within it and having the same axis with it, but exterior to each other. Translation of a vector around one of the boundaries of the plane map is represented by an α-curve on one of the tores; while rotation of a vector at a point through an angle 2π is represented by a β-curve if opposite directions on the surface are treated as distinct, by a β^2-curve if opposite directions are considered identical.

This region R is, according to § 3, homeomorphic with the part of a space of inversion S exterior to $p+1$ tores $T_0, T_1, \cdots, T_p$ of which one, T_0, links all the rest. The homeomorphism carries α-curves into β-curves and β-curves into α-curves on T_0, but does not change the classes of curves on $T_1, \cdots, T_p$.

The states of motion on the other part of the surface are likewise represented by the points of a region R' within a tore and outside of p others; and R' is homeomorphic with a region in a sheet S' covering that in S which corresponds to R.

To see how the tores bounding R are matched with those bounding R' we find by the method of A, § 3, that translation of a vector around a boundary of one of the plane maps is equivalent to motion, with rotation through an angle 4π, around the corresponding boundary of the other map. This presupposes such orientations of the maps that a positive rotation on a boundary of one is equivalent to a negative rotation on the other. Hence the transformation of a boundary of R into a boundary of R' is

$$\alpha' = \alpha\beta^{2\epsilon}, \quad \beta' = \beta^{-1}, \tag{1}$$

where $\epsilon = 1$ if opposite directions are considered distinct and $\epsilon = 2$ if opposite directions are considered identical. Of the boundaries in S, $T_1, \cdots, T_p$ will be transformed respectively into boundaries $T_1', \cdots, T_p'$ of S' in such a way that (1) holds; while for the exceptional tores T_0 and T_0' which link the others the transformation is expressed by interchanging α with β and α' with β' in (1), so that

$$\alpha' = \alpha^{-1}, \quad \beta' = \alpha^{2\epsilon}\beta. \tag{2}$$

But (2) is precisely the transformation Δ_n of § 1, with $n = 2\epsilon$. We therefore join the sheets S and S' along a strip which has an ϵ-fold twist, its boundaries being curves of type $\alpha^{\epsilon}\beta$ on the tores T_0 and T_0'.

The tores $T_1, \cdots, T_p$ bounding the region in S are deformed into the respectively corresponding tores $T_1', \cdots, T_p'$ in S' after interposing two new sheets S_i and S_i' for each T_i. The transformation (1) of T_i into T_i' is equivalent to $\Delta_0 D\Delta_{-2\epsilon} D\Delta_0$. A deformation of this type is possible which sweeps out S_i and S_i' if S and S_i join along a strip having a 0-fold twist,

S_i and S_i' along a strip having a $(-\epsilon)$-fold twist, and S_i' and S' along a strip having a 0-fold twist.

We thus have $2p+2$ sheets. A picture of the complete set of branch curves is obtained, for $p=2$ and $\epsilon=1$, by deleting from the figure on page 489 the four curves at the top.

8. **States of motion on one-sided surfaces.** A non-orientable surface of odd connectivity can, like an orientable surface, be represented by a pair of plane regions, each bounded by circles, with the one difference that some of the circles of one map are to be identified with those of the other in a sense-preserving and some in a sense-reversing manner, whereas in the orientable case the mode of identification is the same for all the circles. If the connectivity of the surface is $2p+1$, the boundary of each map will consist of $p+1$ circles of which the number in each of the two classes is at least one but is otherwise arbitrary.

Thus the manifold of states of motion consists of two regions R and R' of which some of the boundaries match as in the last section and some in a different manner. This matching is determined by a geometrical argument similar to that of *A*, § 3. It may be shown that if a circle of one map matches a circle of the other map in a sense-reversing manner, a vector at a point θ of one circle making an angle φ with a fixed direction in the plane is to be identified with a vector at a point θ' of the other circle making an angle φ' with this fixed direction, where

$$\theta' = -\theta, \qquad \varphi' = \varphi - 2\theta, \tag{3}$$

with an arbitrary constant added in each case. This shows that the transformation connecting the corresponding boundaries of R and R' is of the class

$$\alpha' = \alpha^{-1}\beta^{-2\epsilon}, \quad \beta' = \beta; \tag{4}$$

for variation of θ through 2π with φ constant will, according to (3), cause θ' to vary through -2π and φ' through -4π; while variation of φ with θ constant causes φ' to vary through an equal angle.

Mapping R and R' respectively upon S and S' as before, we see that each of the tores $T_1, \cdots, T_p$ is to be deformed into one of the tores $T_1', \cdots, T_p'$ in such a way that either (1) or (4) holds; while for T_0 and T_0' the transformation is either (2) or that derived from (4) in the same way that (2) is derived from (1); namely

$$\alpha' = \alpha, \quad \beta' = \alpha^{-2\epsilon}\beta^{-1}. \tag{5}$$

Any of these transformations may be effected by deformation through intermediate sheets; for (4) is equivalent to $\Delta_0 D\Delta_0 D\Delta_0 D\Delta_{-2\epsilon} D\Delta_0$, and (5) to $\Delta_0 D\Delta_0 D\Delta_{2\epsilon}$, while it has been seen in § 7 that (1) is equivalent to $\Delta_0 D\Delta_{-2\epsilon} D\Delta_0$ and (2) to $\Delta_{2\epsilon}$. There is also an infinite number of other modes of expressing these transformations in terms of sequences of D's and Δ's alternating, with a Δ on each end. For example, any of these transformations may be followed by any number of transformations such as $\Delta_n D\Delta_0 D\Delta_{-n} D\Delta_0 D$, which is equivalent to the identity.

The simplest of the corresponding covering spaces has $2p+4$ sheets $S, S', S_0, S_1, \cdots, S_p, S_0', S_1', \cdots, S_p'$, of which S is joined to $S_0, S_1, \cdots, S_p$ and S' to $S_1', \cdots, S_p'$ along strips of zero twist; S_0' is joined to S_0 along a strip of zero twist and to S' along a strip of ϵ-fold twist; while S_i $(i=1, \cdots, p)$ is joined to S_i' along a strip of $(-\epsilon)$-fold twist.

9. **Surfaces of even connectivity; $\epsilon=1$.** A surface of even connectivity, necessarily one-sided, is defined* as the sum, modulo 2, of an orientable surface and a projective plane having a 2-cell in common. We may therefore expect to find a covering space representing the states of motion on such a surface by a suitable combination of that of § 7 with a covering space representing the states of motion on a projective plane. A space of the latter kind was found in *A*, §§ 5–8, for the case in which opposite directions are considered distinct, a case which we shall now discuss. This covering space has two sheets which join along a strip having a 2-fold twist (illustrated in *A*, Fig. 3).

Let the 2-cell common to the orientable surface and the projective plane be J and let its boundary be j. The removal of J from either surface implies the removal from the manifold of states of motion of the interior or exterior of a tore T which represents the states of motion on j. In the manifold of § 7 we shall let T be a tore in the sheet S which links the strip joining S and S'. It will then be the *interior* of T that represents states of motion on J. Translation along j will be represented by an α-curve of T, and rotation at a point by a β-curve. Translation is of course defined with reference to a system of curves on J which is the image of a system of parallels in a plane.

In order to combine the two covering spaces we must find in that for the projective plane a region, homeomorphic with the interior of a tore, which represents the states of motion on the 2-cell J. We must also determine what curves on the boundary of this region represent translation of a vector along j, and what curves represent rotation at a point. A curve

* O. Veblen, *Cambridge Colloquium Lectures*, p. 48.

of the first kind can, by taking J sufficiently small, be enclosed by as small a sphere as we please, for the variation in direction of the vector is zero. Such a curve can therefore be deformed to a point in the manifold, and may be taken at will as a curve α or α^{-1}on the tore.

Rotation of a vector at a point of j will be represented on the tore by some curve of type $\alpha^n\beta^{\pm 1}$. We may determine n, as well as the manner of the tore's winding about, by finding how to mark into Fig. 3 of A two curves of which each represents rotation at a point, for these curves will have with each other linkage of order n.

For this purpose we examine a correspondence of the two-sheeted space with the cylindrical region of Fig. 4 of A. A plane s parallel to the bases of this cylinder and half way between them divides it into two regions R_1 and R_2. The points of the curved surface are to be identified in pairs as specified in A, § 5. If this identification be made only for the point pairs of which both points lie on the surface of R_1, this region becomes homeomorphic with the interior of a tore; and similarly for R_2. The regions R_1 and R_2 are then canonical regions (cf. A, § 2). The canonical cuts are represented on the cylindrical surface by four 2-cells, each of which is bounded by an arc on s, an arc on a base of the cylinder, an arc of one of the helices shown in Fig. 4 of A, and an arc of another helix congruent with this one.

It is not difficult to show that a correspondence F between cylindrical region and covering space may be set up under which each of the regions R_1 and R_2 corresponds in 1–1 continuous fashion to one sheet and under which one of the branch curves of Fig. 3, A, which we shall call γ, is the correspondent of the two generators of the cylinder shown in Fig. 4, A; these we shall call γ' and $\overline{\gamma}'$. The double strip bounded by the branch curves, along which the sheets join, is of course the correspondent under F of the common surface of R_1 and R_2.

Regarding Fig. 3, A, as a projection of the branch curves on a plane, we may consider that the four narrow regions in that figure, each bounded by two arcs, represent the strip bounded by the branch curves, while the central four-sided region represents the two canonical cuts, one in each sheet.

Let γ_1' and γ_2' be two lines within the cylinder parallel to γ', and let γ_1 and γ_2 be curves in the covering space which are the respective correspondents under F of γ_1' and γ_2'. Each of the curves γ', γ_1', γ_2', and therefore γ, γ_1, γ_2, represents rotation of a vector at a point. But while γ' and $\overline{\gamma}'$ lie entirely on the common surface of R_1 and R_2, γ_1' and γ_2' consist each of two segments, one in R_1 and one in R_2.

Since γ_1' has no point in common with either canonical cut, γ_1 may be projected upon Fig. 3, A, in such a way that the projected curve has no point within the central four-sided region. Therefore, since γ_1 cuts only twice the strip bounded by γ and the other branch curve, and since γ_1 may be taken arbitrarily close to γ, it cannot link γ.

In the same way it may be shown that γ_2 does not link γ. From a consideration of the relations of γ_1', γ_2' and γ' it will be seen further that γ_1 is deformable to γ through a set of intermediate positions containing no point of γ_2. *Hence γ_1 does not link γ_2.*

Thus $n=0$, and rotation of a vector at a point of j is represented by a β-curve on the tore in the two-sheeted covering space whose interior represents states of motion on J. The matching of this tore with the tore T, of the covering space representing states of motion on the orientable surface, is determined topologically simply by

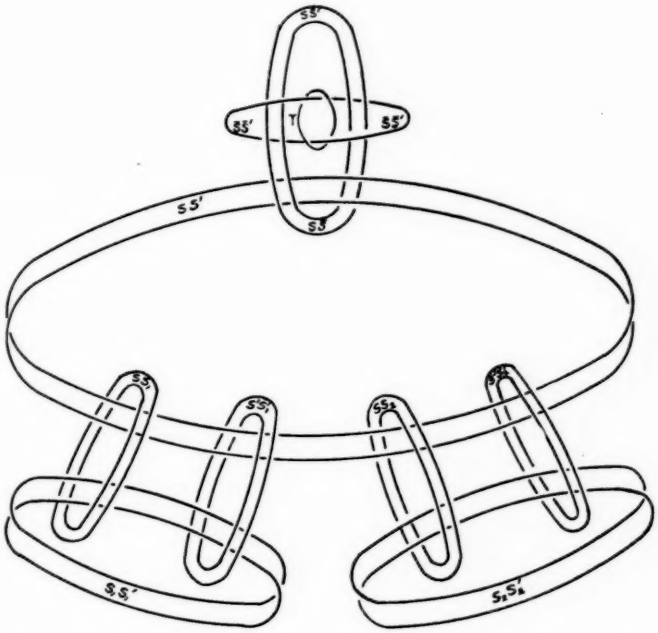
$$\alpha' = \alpha^{\pm 1}, \quad \beta' = \beta^{\pm 1},$$

where we may take the signs as we please. Hence the two spaces are to join along a strip of zero twist bounded by curves such as γ_1 and γ_2. Denote the sheets of the two-sheeted space by $\overline{S}$ and $\overline{S}'$. Each of these sheets will join S along a part of the strip just mentioned, for it pierces twice the strip along which $\overline{S}$ and $\overline{S}'$ join each other.

The system of branch curves for a covering space representing states of motion on a surface of connectivity 6 is shown in the figure on page 489.

10. The manifolds of states of motion on surfaces of even connectivity with opposite directions treated as identical are more complicated and can apparently be represented as covering spaces only by a rather laborious process. As it is not clear that much can be gained in applications by requiring opposite directions to be identified, we shall give this case very summary treatment.

The manifold of states of motion on a projective plane is in this case represented by a cylindrical region of height π. The two bases are to be identified in the manner determined by a translation, and the point of the curved surface whose cylindrical coördinates (the axis of the coördinates being the axis of the cylinder) are (r, θ, φ) is to be identified with the point $(r, \theta+\pi, 2\theta-\varphi)$, the last coördinate being reduced modulo π. (In the case of distinct opposite directions the reduction was modulo 2π.) A plane s parallel to the bases of the cylinder and half way between them divides the interior into two regions. Two 2-cells on the boundary of each of these regions are to be identified respectively in the manner just indicated with two other 2-cells on the boundary of the same region. In this way a Heegaard diagram of genus 2 is obtained.


S S'
S S'
S S'

It is noteworthy that a manifold of the states of motion in an n-dimensional manifold, while not in general a product manifold itself, may be divided into regions of which each is the product of an n-cell by an $(n-1)$-dimensional sphere (if opposite directions are considered distinct) or projective space (if opposite directions are treated as identical).

The only closed product manifolds which are three-dimensional covering spaces have as factors a closed curve and an orientable surface; for the product of a curve and a one-sided surface is one-sided; while a covering space is necessarily orientable, because a curve in it along which the indicatrix were reversed would have to cover such a curve in the orientable fundamental space.

As in § 7 let an orientable surface of genus p be mapped on two plane regions, each bounded by $p+1$ circles. The product of each region by a closed curve is, as in § 7, homeomorphic with the space exterior to $p+1$ tores of which one links all the rest. But the manner of joining the sheets is simpler in the present case; for we may set up the homeomorphism in such a way that an α-curve on each tore in the first sheet is to be identified with an α^{-1}-curve on the corresponding tore in the second sheet, and a β-curve with a β-curve. From § 1 it then follows that the manifold is homeomorphic with a two-sheeted space having $2p+2$ knotless branch curves of which two link all the rest in the simplest fashion, but which do not link otherwise.

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A THEORY OF A GENERAL NET ON A SURFACE*

BY

VERNON G. GROVE

1. Introduction

The purpose of this investigation is to carry on the researches of G. M. Green concerning the projective differential geometry of nets of space curves. We assume that the net is not a conjugate net, and, unless otherwise stated, that it is not the asymptotic net. We assume, furthermore, that the sustaining surface is non-developable. The present investigation may, therefore, be considered as an extension of Green's paper on *Nets of space curves.*†

We summarize for future reference some of the results obtained by Green in the paper cited. Let the homogeneous coördinates $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ of a point in projective space of three dimensions be given as analytic functions

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4) \tag{1}$$

of two variables u and v. Let C_u denote the curve defined by $v = \text{const.}$, and C_v the curve defined by $u = \text{const.}$ The net of curves C_u, C_v lies on a surface S_y. If this net is not a conjugate net, the four functions $y^{(k)}(u, v)$ satisfy a system of partial differential equations of the form

$$\begin{aligned} y_{uu} &= a\, y_{uv} + b\, y_u + c\, y_v + d\, y, \\ y_{vv} &= a' y_{uv} + b' y_u + c' y_v + d' y \qquad (1 - aa' \neq 0). \end{aligned} \tag{2}$$

The coefficients of system (2) are related by four conditions of complete integrability, one of which‡ is

$$a_u^{(12)} + c_u' = a_v^{(21)} + b_v, \tag{3}$$

wherein $a^{(21)}$ and $^{(12)}$ are defined by the formulas

$$\begin{aligned} (1 - aa')a^{(21)} &= a_v + a'c + b + a(a_u' + ab' + c'), \\ (1 - aa')a^{(12)} &= a_u' + ab' + c' + a'(a_v + a'c + b). \end{aligned} \tag{4}$$

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* Presented to the Society, December 26, 1924; received by the editors in December, 1925.

† G. M. Green, *Nets of space curves*, these Transactions, vol. 21 (1920), pp. 207–236. Hereafter referred to as *Nets*.

‡ *Nets*, p. 212.

Equation (3) shows the existence of a function $f(u,v)$, such that

$$f_u = a^{(21)} + b, \qquad f_v = a^{(12)} + c'. \tag{5}$$

In case the parametric net is not asymptotic, so that $aa' \neq 0$,* we find it convenient to write

$$\alpha = \frac{1}{a}, \quad \beta = -\frac{b}{a}, \quad \gamma = -\frac{c}{a}, \quad \delta = -\frac{d}{a},$$

$$\alpha' = \frac{1}{a'}, \quad \beta' = -\frac{b'}{a'}, \quad \gamma' = -\frac{c'}{a'}, \quad \delta' = -\frac{d'}{a'}.$$

Each of the points ρ and σ defined by

$$\rho = y_u - \gamma y, \qquad \sigma = y_v - \beta' y \tag{6}$$

is the focal point of the parametric tangent on which it lies. The line joining the points ρ and σ is called the *ray of the point* y, or the *parametric ray*. The line of intersection of the osculating planes at y of the curves C_u and C_v is called the *axis of the point* y, or the *parametric axis*. The harmonic conjugate of y with respect to the focal points of the axis is the point τ defined by the expression

$$\tau = y_{uv} - \beta' y_u - \gamma y_v - \tfrac{1}{2} My, \tag{7}$$

where

$$M = b^{(21)} + c^{(12)} + \gamma a^{(12)} + \beta' a^{(21)} - b\beta' - c'\gamma - a\beta'^2 - a'\gamma^2 - 2\beta'\gamma - \beta_u' - \gamma_v,$$

the functions $b^{(21)}$, $c^{(12)}$ being defined by the formulas

$$\begin{aligned} (1 - aa')b^{(21)} &= b_v + b'c + a(bb' + b_u' + d'), \\ (1 - aa')c^{(12)} &= c_u' + b'c + a'(cc' + c_v + d). \end{aligned} \tag{8}$$

2. The relation R between two congruences

The concept of congruences in relation R with respect to the asymptotic net has been treated by Green.† He pointed‡ out that the concept can be also applied to any non-conjugate net. We will discuss that relation with respect to a net which is not conjugate.

Let $R^{(u)}$ be the ruled surface formed by the tangents to the curves C_u at the points where they meet the curve C_v through y. Similarly let $R^{(v)}$

* E. J. Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), p. 243.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 86. Hereafter referred to as *Surfaces*.

‡ *Surfaces*, footnote p. 87.

be the ruled surface formed by the tangents to C_v. Let r be a point on the tangent to C_u at y, and s be a point on the tangent to C_v at y. The tangent planes at r and s, to $R^{(u)}$ and $R^{(v)}$ respectively, intersect in a line l' through the point y but not lying in the tangent plane to S_y at y. The line l joining r and s and the line l' are said to be in *relation R*. It may readily be verified that the line l joining the points

$$r = y_u - \lambda y, \qquad s = y_v - \mu y, \tag{9}$$

and the line l' joining y to the point

$$z = y_{uv} - \mu y_u - \lambda y_v, \tag{10}$$

where λ and μ are arbitrary functions of u and v, are lines in relation R. For given functions λ, μ, the lines l and l' generate congruences Γ and Γ'; Γ and Γ' are said to be *congruences in relation R*.

We see, from (6) and (7), that *the ray and axis of y are in relation R*.

3. The developables and focal surfaces of Γ and Γ'

To every curve $v=v(u)$ on S_y there corresponds a ruled surface of Γ. We proceed to find the differential equation satisfied by $v(u)$ in order that this ruled surface be developable. Any point P on l may be written

$$P = r + \theta s.$$

As y moves along $v=v(u)$, P describes a curve. The point dP/du lies on the tangent to this curve at P. We find that

$$\begin{aligned}\frac{dP}{du} = &\left[a+\theta+(1+a'\theta)\frac{dv}{du}\right]y_{uv} + \left\{d-\lambda_u-\mu_u\theta+(d'\theta-\mu_v\theta-\lambda_v)\frac{dv}{du}\right.\\ &\left.+\lambda\left(b-\lambda-\mu\theta+b'\theta\frac{dv}{du}\right)+\mu\left[c+(c'\theta-\lambda-\mu\theta)\frac{dv}{du}\right]\right\}y+(\ )r+(\ )s,\end{aligned} \tag{11}$$

the coefficients of r and s being immaterial. The curve $v=v(u)$ therefore corresponds to a developable of Γ if, and only if, the coefficients of y_{uv} and y in (11) vanish. Eliminating θ from the equations so obtained we find the following differential equation of the curve $v=v(u)$:

$$\begin{aligned}&[d - \lambda_u + \lambda(b - \lambda) + c\mu + a\lambda\mu + a\mu_u]du^2\\ &+\{[a'(d - \lambda_u) + a'\lambda(b - \lambda) + a'c\mu + \mu_u]\\ &-[a(d' - \mu_v) + a\mu(c' - \mu) + ab'\lambda + \lambda_v]\}dudv\\ &-[d' - \mu_v + \mu(c' - \mu) + b'\lambda + a'\lambda\mu + a'\lambda_v]dv^2 = 0.\end{aligned} \tag{12}$$

The curves (12) will be referred to as the Γ-*curves*.

If we eliminate dv/du from the coefficients of y_{uv} and y in (11) equated to zero, we find the condition on θ that P be the point of contact of the line l with the edge of regression of the developable to which it belongs. We find that the focal points on l are defined by the roots of the quadratic

$$\begin{aligned}(13)\qquad &[d' - \mu_v + \mu(c' - \mu) + b'\lambda + a'\lambda\mu + a'\mu_u]\theta^2 \\ &+ \{[a(d' - \mu_v) + a\mu(c' - \mu) + ab'\lambda + \mu_u] \\ &- [a'(d - \lambda_u) + a'\lambda(b - \lambda) + a'c\mu + \lambda_v]\}\theta \\ &- [d - \lambda_u + \lambda(b - \lambda) + c\mu + a\lambda\mu + a\lambda_v] = 0.\end{aligned}$$

In a similar way we find that the developables of the Γ' congruence cut S_y in the curves whose differential equation is

$$\begin{aligned}(14)\qquad &[c^{(21)} - c\mu - \lambda_u + \lambda(a^{(21)} - a\mu - \lambda)]du^2 \\ &+ \{[c^{(12)} - c'\lambda - \lambda_v + \lambda(a^{(12)} - a'\lambda - \mu)] \\ &- [b^{(21)} - b\mu - \mu_u + \mu(a^{(21)} - a\mu - \lambda)]\}dudv \\ &- [b^{(12)} - b'\lambda - \mu_v + \mu(a^{(12)} - a'\lambda - \mu)]dv^2 = 0.\end{aligned}$$

The curves (14) will be called the Γ'-*curves*. We find also that the focal points of l' are defined by the expressions

$$z_1 = z + \varphi_1 y, \qquad z_2 = z + \varphi_2 y,$$

where φ_1, φ_2 are the roots of the quadratic

$$\begin{aligned}(15)\qquad &\varphi^2 + [c^{(12)} - c'\lambda - \lambda_v + b^{(21)} - b\mu - \mu_u + \lambda(a^{(12)} - a'\lambda - \mu) + \mu(a^{(21)} - a\mu - \lambda)]\varphi \\ &+ [(b^{(21)} - b\mu - \mu_u)(c^{(12)} - c'\lambda - \lambda_v) - \lambda(a^{(21)} - a\mu - \lambda)(b^{(12)} - b'\lambda - \mu_v) \\ &- (c^{(12)} - c\mu - \lambda_u)(b^{(12)} - b'\lambda - \mu_v) - \mu(a^{(12)} - a'\lambda - \mu)(c^{(21)} - c\mu - \lambda_u) \\ &+ \lambda(a^{(12)} - a'\lambda - \mu)(b^{(21)} - b\mu - \mu_u) + \mu(a^{(21)} - a\mu - \lambda)(c^{(12)} - c'\lambda - \lambda_v)] = 0.\end{aligned}$$

The harmonic conjugate of y with respect to z_1 and z_2 is defined by the formula.

$$\begin{aligned}(16)\qquad Z = z - \tfrac{1}{2}&[c^{(12)} - c'\lambda - \lambda_v + b^{(21)} - b\mu - \mu_u \\ &+ \lambda(a^{(12)} - a'\lambda - \mu) + \mu(a^{(21)} - a\mu - \lambda)]y.\end{aligned}$$

The harmonic invariant of the quadratic (12) and the differential equation of the asymptotic curves*

$$adu^2 + 2dudv + a'dv^2 = 0$$

is

$$(17)\qquad \mu_u - \lambda_v.$$

* *Nets*, p. 213.

Hence if with Green* we call those congruences Γ, whose developables correspond to a conjugate net on S_y, *harmonic to the surface*, we see that the Γ-*congruence is harmonic to the surface if and only if the expression* (17) *vanishes.*†

Green has called those congruences Γ' whose developables cut S_y in a conjugate net *conjugate to the surface.* We find that *the* Γ' *congruence is conjugate to* S_y *if and only if*

$$\frac{\partial}{\partial u}(c' + a'\lambda + \mu) = \frac{\partial}{\partial v}(b + a\mu + \lambda). \tag{18}$$

4. Some general properties of congruences in relation R

To the curves C_u and C_v there correspond two ruled surfaces $S^{(u)}$ and $S^{(v)}$ of the Γ' congruence. If C_v does not coincide with a Γ'-curve, the plane determined by l' and the tangent to C_v at y is tangent to $S^{(u)}$ at a definite point ζ_1 of l'. This point must be given by an expression of the form

$$\zeta_1 = z + \omega y.$$

By imposing the condition that ζ_{1u} must lie in this plane, we may determine ω. We find that the plane determined by the tangent to C_v and l' touches $S^{(v)}$ in the point

$$\zeta_1 = z - [b^{(21)} - b\mu - \mu_u + \mu(a^{(21)} - a\mu - \lambda)]y. \tag{19}$$

Similarly the plane determined by the tangent to C_u at y and l' touches $S^{(v)}$ at the point

$$\zeta_2 = z - [c^{(12)} - c'\lambda - \lambda_v + \lambda(a^{(12)} - a'\lambda - \mu)]y. \tag{20}$$

The totality of points r defined by (9) form a surface S_r. The tangent at r to the curve $u = \text{const.}$ on S_r meets l' in the point

$$\eta_1 = z - (\lambda_v - \lambda\mu)y. \tag{21}$$

Similarly the tangent at s to the curve $v = \text{const.}$ on S_s meets l' in the point

$$\eta_2 = z - (\mu_u - \lambda\mu)y. \tag{22}$$

The points η_1 *and* η_2 *coincide if and only if* $\mu_u - \lambda_v = 0$, *that is, if and only if the* Γ *congruence is harmonic to the surface. The points* ζ_1 *and* ζ_2 *coincide if and only if the* Γ' *tangents separate the parametric tangents harmonically.*

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* *Surfaces*, p. 99.

† It is verified readily that, whatever parametric net is used, the line joining $r = y_u - \lambda y$, $s = y_v - \mu y$ is harmonic to the surface if $\mu_u - \lambda_v = 0$.

5. The osculating quadrics of the parametric ruled surfaces of tangents

By methods similar to those used by Green,* we find that the quadrics osculating the parametric ruled surfaces of tangents $R^{(u)}$ and $R^{(v)}$ along their generators through y are respectively

$$(23)\qquad \begin{aligned} Q^{(u)} &= \tfrac{1}{2}a'x_3^2 - \tfrac{1}{2}c^{(12)}x_4^2 - x_1x_4 + x_2x_3 + \tfrac{1}{2}(f_v - 2c')x_3x_4 = 0, \\ Q^{(v)} &= \tfrac{1}{2}ax_2^2 - \tfrac{1}{2}b^{(21)}x_4^2 - x_1x_4 + x_2x_3 + \tfrac{1}{2}(f_u - 2b)x_2x_4 = 0, \end{aligned}$$

the tetrahedron of reference being y, y_u, y_v, y_{uv}.

By comparing $Q^{(u)}$ and $Q^{(v)}$, we find that *$Q^{(u)}$ and $Q^{(v)}$ coincide if and only if $a = a' = 0$, that is, if and only if C_u and C_v are asymptotic curves.* Since the coördinates of ρ defined by (6) are $(\gamma, -1, 0, 0)$, we see that *ρ lies on $Q^{(u)}$, but never on $Q^{(v)}$*; similarly *σ lies on $Q^{(v)}$ but never on $Q^{(u)}$. The polar plane of τ, defined by* (7), *with respect to $Q^{(u)}$ passes through ρ and will pass through σ if and only if the invariant*

$$(24)\qquad \mathfrak{B} = a'\gamma - \tfrac{1}{2}(f_v - 2c')$$

vanishes. The polar plane of τ with respect to $Q^{(v)}$ passes through σ and will pass through ρ if and only if the invariant

$$(25)\qquad \mathfrak{C}' = a\beta' - \tfrac{1}{2}(f_u - 2b)$$

vanishes.

The line l' cuts $Q^{(u)}$ and $Q^{(v)}$ in y and the points $Z^{(u)}$ and $Z^{(v)}$ respectively, where

$$(26)\qquad \begin{aligned} Z^{(u)} &= z + \tfrac{1}{2}[2\lambda\mu - \lambda(a^{(12)} - c' - a'\lambda) + c^{(12)}]y, \\ Z^{(v)} &= z + \tfrac{1}{2}[2\lambda\mu - \mu(a^{(21)} - b - a\mu) + b^{(21)}]y. \end{aligned}$$

The point $Z^{(u)}$ is the harmonic conjugate of y with respect to η_1, and ζ_2 defined by (21) *and* (20); *$Z^{(v)}$ is the harmonic conjugate of y with respect to η_2 and ζ_1 defined by* (22) *and* (19).

6. The R-reciprocal congruences

The line l intersects $Q^{(u)}$ in the points r and $P^{(u)}$ defined by

$$P^{(u)} = a'r - 2s$$

and intersects $Q^{(v)}$ in s and $P^{(v)}$,

$$P^{(v)} = as - 2r.$$

The equations of the tangent planes to $Q^{(u)}$ at r and $P^{(u)}$ are respectively

$$x_3 + \lambda x_4 = 0, \quad 2x_2 + a'x_3 + 2(\mu - a'\lambda + f_v - 2c')x_4 = 0.$$

* *Surfaces*, pp. 96, 97.

The points y and $(0, \mu+\frac{1}{2}f_v-c'-a'\lambda, \lambda, -1)$ lie on the intersection of these planes. Therefore the polar line $l^{(u)}$ of l with respect to $Q^{(u)}$ joins y to the point defined by

$$\zeta^{(u)} = y_{uv} - (\mu - \tfrac{1}{2}f_v - c' - a'\lambda)y_u - \lambda y_v.$$

Similarly, the polar line $l^{(v)}$ of l with respect to $Q^{(v)}$ joins y to the point

$$\zeta^{(v)} = y_{uv} - \mu y_u - (\lambda + \tfrac{1}{2}f_u - b - a\mu)y_v.$$

The lines $l^{(u)}$ and $l^{(v)}$ coincide if and only if

$$\lambda = \frac{1}{2a'}(f_v - 2c'), \quad \mu = \frac{1}{2a}(f_u - 2b). \tag{27}$$

Hence the line l joining

$$r = y_u - \frac{1}{2a'}(f_v - 2c')y, \quad s = y_v - \frac{1}{2a}(f_u - 2b)y \tag{28}$$

has coincident reciprocal polars with respect to $Q^{(u)}$ and $Q^{(v)}$; this reciprocal polar joins y to the point Z defined by

$$Z = y_{uv} - \frac{1}{2a}(f_u - 2b)y_u - \frac{1}{2a'}(f_v - 2c')y_v. \tag{29}$$

Evidently this unique pair of reciprocal polars are in relation R; we will call them the *R-reciprocal lines*, and the congruences generated by them the *R-reciprocal congruences*.

It follows from (28) and (6) that r will coincide with ρ if and only if $\mathfrak{B}=0$; and s will coincide with σ if and only if $\mathfrak{C}'=0$. Hence *the R-reciprocal congruences coincide with the ray and axis congruences if and only if* $\mathfrak{B}=\mathfrak{C}'=0$.

The quadrics $Q^{(u)}$ and $Q^{(v)}$ intersect in a nodal quartic with node at y. The projecting cone of this quartic from y is a quadric cone K. As l' generates K, the line l in relation R to l' envelops a conic C. *The polar line of y with respect to C is the line joining rs. The polar planes of the focal points ρ and σ with respect to K intersect in the line yZ.* If K degenerates into two distinct planes, the line yZ is their intersection. In this case C degenerates into the line rs counted twice.

7. The associate conjugate net

The net defined by the differential equation

$$adu^2 - a'dv^2 = 0 \tag{30}$$

has been called the *associate conjugate net*.* The tangents at y to the curves

* *Nets*, p. 213

of this net are the double rays of the involution determined by the parametric tangents and the asymptotic tangents.

Let us call the line joining the focal points of the associate conjugate tangents *the associate ray*, and the line of intersection of the osculating planes at y of the associate conjugate curves *the associate axis*. Any point on the tangent to the curve defined by $dv/du=m$, $m=\sqrt{a/a'}$ is defined by an expression of the form

$$\bar{\rho} = y_u + my_v + \omega_1 y. \tag{31}$$

By methods similar to those used in § 3, we determine ω_1 so that $\bar{\rho}$ is the focal point of the tangent on which it lies. We obtain

$$\omega_1 = \frac{(1+a'm)(c+m_u-bm)-(a+m)(c'm+m_v-b'm^2)}{a+2m+a'm^2}. \tag{32}$$

By changing the sign of m in (31) and (32), we find that the focal point of the tangent to the curve defined by $dv/du=-m$ is the point

$$\bar{\sigma} = y_u - my_v + \omega_2 y, \tag{33}$$

where

$$\omega_2 = \frac{(1-a'm)(c-m_u+bm)+(a-m)(c'm+m_v+b'm^2)}{a-2m+a'm^2}. \tag{34}$$

It may be verified from (31) and (33) that the associate ray intersects the parametric tangents in the points

$$\begin{aligned} r &= y_u - \frac{1}{2}\left(b - b'm^2 - \frac{m_u}{m}\right) y, \\ s &= y_v - \frac{1}{2}\left(c' - cm'^2 - \frac{m_v'}{m'}\right) y, \qquad mm' = 1. \end{aligned} \tag{35}$$

By comparing (6) with (35), we find that r will coincide with ρ if and only if the invariant

$$\mathfrak{S} = b - b'm^2 - 2\gamma - \frac{m_u}{m} \tag{36}$$

vanishes. Similarly s will coincide with σ if and only if the invariant

$$\mathfrak{S}' = c' - cm'^2 - 2\beta' - \frac{m_v'}{m'} \tag{37}$$

vanishes. By comparing (6) and (28), we find that the associate ray coincides with the line rs if and only if the invariants

$$a'\mathfrak{S} + 2\mathfrak{B}, \qquad a\mathfrak{S}' + 2\mathfrak{C}' \tag{38}$$

vanish.

The associate axis joins y to a point defined by an expression of the form

$$z = y_{uv} - \mu y_u - \lambda y_v. \tag{39}$$

Consider first the curve defined by $dv/du = m$. The osculating plane at y to this curve is determined by the points y

$$\frac{dy}{du} = y_u + m y_v,$$

$$\frac{d^2y}{du^2} = (a + 2m + a'm^2)y_{uv} + (b + b'm^2)y_u + (c + m_u + c'm^2 + mm_v)y_v + ( )y,$$

the coefficient of y being immaterial. If we impose the condition that yz shall lie in this plane, we find that λ and μ satisfy the equation

$$(\lambda - m\mu)(a + 2m + a'm^2) = m(b + b'm^2) - c - c'm^2 - mm_v - m_u. \tag{40}$$

By changing the sign of m in (40), we find the condition that yz shall lie in the osculating plane at y to the curve defined by $dv/du = -m$ is

$$(\lambda + m\mu)(a - 2m + a'm^2) = -m(b + b'm^2) - c - c'm^2 - mm_v + m_u. \tag{41}$$

Solving (40) and (41) for λ and μ, we find that the associate axis joins y to the point

$$z = y_{uv} - \mu y_u - \lambda y_v, \tag{39 bis}$$

where

$$\begin{aligned} \lambda &= \frac{1}{2(1 - aa')}\left(b + b'm^2 + ac' + a'c - a\frac{m_v'}{m'} + \frac{m_u'}{m'}\right), \\ \mu &= \frac{1}{2(1 - aa')}\left(c' + cm'^2 + a'b + ab' - a'\frac{m_u}{m} + \frac{m_v}{m}\right). \end{aligned} \tag{42}$$

Equations (7) and (42) show that the associate axis lies in the osculating plane to C_v if and only if the invariant

$$\mathfrak{C} = c' + cm'^2 + a'b - ab' - 2\beta' - a'\frac{m_u}{m} + \frac{m_v}{m} = a'\mathfrak{S} + \mathfrak{S}' \tag{43}$$

vanishes; it lies in the osculating plane to C_u if and only if the invariant

$$(44) \qquad \mathfrak{B}' = b + b'm^2 + ac' - a'c - 2\gamma - a\frac{m_v'}{m'} + \frac{m_u'}{m'} = a\mathfrak{S}' + \mathfrak{S}$$

vanishes. From (43) and (44) it follows that *the associate ray congruence coincides with the parametric ray congruence if, and only if, the associate axis congruence coincides with the parametric axis congruence.*

An interesting special case of this theorem arises in case the parametric tangents form a constant cross ratio with the asymptotic tangents. We may readily verify that the cross ratio formed by these tangents is constant if $aa' = \text{const.}$ Using the formulas for $\{\alpha'\}$ and $\{\beta'\}$ derived by Green,* we find, except for a factor, that

$$(45) \qquad \begin{aligned} \{\alpha'\} &= a'\left[aa'\mathfrak{B}' - \frac{1}{2}\frac{\partial}{\partial u}(aa')\right] - (1 - \sqrt{1 - aa'})\left[aa'\mathfrak{C} - \frac{1}{2}\frac{\partial}{\partial v}(aa')\right], \\ \{\beta\} &= a\left[aa'\mathfrak{C} - \frac{1}{2}\frac{\partial}{\partial v}(aa')\right] - (1 - \sqrt{1 - aa'})\left[aa'\mathfrak{B}' - \frac{1}{2}\frac{\partial}{\partial u}(aa')\right]. \end{aligned}$$

The surface S_v is therefore a quadric if, and only if,

$$(46) \qquad aa'\mathfrak{B}' - \frac{1}{2}\frac{\partial}{\partial u}(aa') = 0, \qquad aa'\mathfrak{C} - \frac{1}{2}\frac{\partial}{\partial v}(aa') = 0.$$

In view of a theorem due to Green,† we may state that *for any net, conjugate or not-conjugate, whose tangents form with the asymptotic tangents a constant cross ratio, the associate axis coincides with the axis of the net if and only if the surface is a quadric.*

Comparing equations (35) and (39), we find that *the associate ray and associate axis are in relation R if and only if* $\mathfrak{B} = \mathfrak{C} = 0$, *that is, if and only if they coincide respectively with the parametric ray and axis.* They will be the R-reciprocal lines if and only if in addition the invariants (38) vanish. Under these conditions, we find that $aa' = \text{const.}$ and the invariants (46) vanish. Hence the *associate ray and associate axis congruences are the R-reciprocal congruences if and only if the sustaining surface is a quadric, and the parameter tangents form a constant cross ratio with the asymptotic tangents.*

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* G. M. Green, *On the theory of curved surfaces and canonical systems in projective differential geometry*, these Transactions, vol. 16 (1915), p. 5, formulas (7).

† G. M. Green, *Projective differential geometry of one-parameter families of space curves*, Second Memoir, American Journal of Mathematics, vol. 38 (1916), p. 317.

8. The inverted associate net

We shall call the net whose differential equation is

$$adu^2 + a'dv^2 = 0, \tag{47}$$

the *inverted associate net.* The tangents to the curves of this net are the double rays of the involution determined by the parametric and the associate conjugate tangents. We shall call the line joining the focal points of the tangents at y to the inverted associate curves the *inverted ray.* By changing m to $n = 1/n' = \sqrt{-a/a'}$ in (31), (32), (33), and (34), we find that the inverted ray cuts the parametric tangents in the points

$$\begin{aligned} r &= y_u - \frac{1}{2}\left(b - a'c + ac' - b'n^2 + \frac{n'_u}{n'} - a\frac{n_v'}{n'}\right)y, \\ s &= y_v - \frac{1}{2}\left(c' - ab' + a'b - cn'^2 + \frac{n_v}{n} - a'\frac{n_u}{n}\right)y. \end{aligned} \tag{48}$$

Comparing (6) and (48), we find that r will coincide with ρ if and only if $\mathfrak{B}'$ vanishes, and s will coincide with σ if and only if $\mathfrak{C}$ vanishes.

The line of intersection of the osculating planes at y to the inverted associate curves will be called the *inverted axis.* By changing m to n in (40) and (41), we find that the inverted axis joins y to the point

$$z = y_{uv} - \frac{1}{2}\left(cn'^2 + c' - \frac{n_v'}{n'}\right)y_u - \frac{1}{2}\left(b'n^2 + b - \frac{n_u}{n}\right)y_v. \tag{49}$$

From equations (8) and (50) we find that *the inverted associate axis lies in the osculating plane to C_v if and only if the invariant $\mathfrak{S}'$ vanishes; it lies in the osculating plane to C_u if and only if $\mathfrak{S}$ vanishes.* We note, from (35) and (48), that *the associate ray and inverted axis are in relation R.* They will be R-reciprocal lines if, and only if, the invariants (38) vanish.

We may state the results of this section as follows: *The inverted axis lies in the osculating plane of a parametric curve if and only if the associate ray meets the tangent to that curve in the focal point. If the associate ray congruence coincides with the parametric ray congruence then the inverted axis congruence coincides with the parametric axis congruence and conversely. If any two of the three rays (axes) coincide then all three rays (axes) coincide and all three axes (rays) coincide. If any two of the three ray (axis) congruences coincide with the plane (space) component of the R-reciprocal congruences, then the surface is a quadric, and the parametric tangents form a constant cross ratio with the asymptotic tangents.*

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CUBIC CURVES AND DESMIC SURFACES*

BY

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CUBIC CURVES AND HESSE'S CONFIGURATION (§ 1–3)

1. **Hesse's configuration.** Let a line $\mathcal{L}$ cut a cubic curve C_6^3 of the sixth class in points A, B, C. The four tangents from A to the curve will have points of contact A_i, $i=0, 1, 2, 3$. Similarly, from B and C we obtain sets B_i and C_i.

Hesse proved (1†) that these 12 points lie by threes on 16 lines with four through each point making a configuration $(12_4, 16_3)$. The notation can be arranged so that we have the following table of collinear points:

$$
(1)\qquad
\begin{array}{llll}
A_0B_0C_0, & A_1B_0C_1, & A_2B_0C_2, & A_3B_0C_3,\\
A_0B_1C_1, & A_1B_1C_0, & A_2B_1C_3, & A_3B_1C_2,\\
A_0B_2C_2, & A_1B_2C_3, & A_2B_2C_0, & A_3B_2C_1,\\
A_0B_3C_3, & A_1B_3C_2, & A_2B_3C_1, & A_3B_3C_0\,.
\end{array}
$$

The notation conforms to the following rules. Three collinear points (1) have different letters; (2) if the subscript 0 occurs, the other two subscripts are alike; (3) when 0 does not occur, the three subscripts are all different.

The four points with the same letter are "cotangential," that is, the tangents to the cubic at these points are concurrent at a point on the curve. Four cotangential points have also been called "corresponding points" (2† Salmon, Cayley) and "conjugate points" (4† Schroeter). An important property of such a set is that their tangential point and the vertices of the diagonal triangle of their quadrangle constitute another such set on the same cubic.

2. **Some known properties of the configuration.** This configuration of Hesse's has been investigated frequently. We proceed to restate some of its properties in preparation for some further developments. Detailed proofs and references can be found in Schroeter's article (5†).

i. The outstanding characteristic of the configuration is that the 12 points are the vertices of three quadrangles any two of which are in four-fold perspective from the vertices of the third.

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† These numbers refer to the papers listed at the end of the article.

ii. Such a configuration can be constructed linearly from two perspective triangles; and there is a unique cubic curve which circumscribes it.

iii. Every configuration of Hesse determines a conjugate configuration of the same kind. Indeed, the *sides* of the quadrangles $\{A\}$, $\{B\}$, $\{C\}$ meet in triples in 12 points of such a figure, as may be briefly indicated as follows. The table (1) gives the following central perspectivities for three triangles:

$$A_0 \left| \begin{matrix} B_1B_2B_3 \\ C_1C_2C_3 \end{matrix} \right. \qquad B_0 \left| \begin{matrix} C_1C_2C_3 \\ A_1A_2A_3 \end{matrix} \right. \qquad C_0 \left| \begin{matrix} A_1A_2A_3 \\ B_1B_2B_3 \end{matrix} \right. .$$

When three triangles are centrally perspective in pairs at three collinear points, they are axially perspective on one line (3* p. 379). Therefore

$$\left| \begin{matrix} B_1B_2 \\ C_1C_2 \\ A_1A_2 \end{matrix} \right| X_1 \qquad \left| \begin{matrix} B_2B_3 \\ C_2C_3 \\ A_2A_3 \end{matrix} \right| Z_3 \qquad \left| \begin{matrix} B_3B_1 \\ C_3C_1 \\ A_3A_1 \end{matrix} \right| Y_2$$

with X_1, Y_2, Z_3 collinear. The totality of all such triples of perspective triangles in the figure gives only 12 such points on account of the property of four-fold perspectivity. The derived points are in three sets $\{X_i\}$, $\{Y_i\}$, $\{Z_i\}$, $i=0, 1, 2, 3$; and their positions are given specifically in the tables which follow.

(2)

	B_0B_1	B_0B_2	B_0B_3	B_1B_2	B_1B_3	B_2B_3
A_0A_1	Z_0					Z_1
A_0A_2		Y_1			Y_0	
A_0A_3			X_2	X_3		
A_1A_2			X_0	X_1		
A_1A_3		Y_3			Y_2	
A_2A_3	Z_2					Z_3

	A_0A_1	A_0A_2	A_0A_3	A_1A_2	A_1A_3	A_2A_3
C_0C_1	Z_1					Z_2
C_0C_2		Y_0			Y_3	
C_0C_3			X_3	X_0		
C_1C_2			X_2	X_1		
C_1C_3		Y_1			Y_2	
C_2C_3	Z_0					Z_3

* This number refers to the papers listed at the end of the article.

	B_0B_1	B_0B_2	B_0B_3	B_1B_2	B_1B_3	B_2B_3
C_0C_1	Z_2					Z_1
C_0C_2		Y_3			Y_0	
C_0C_3			X_0	X_3		
C_1C_2			X_2	X_1		
C_1C_3		Y_1			Y_2	
C_2C_3	Z_0					Z_3

Thus X_0 is the intersection of A_1A_2, B_0B_3, C_0C_3.

iv. The 12 derived points lie in pairs on the sides of the complete quadrangles $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ and each pair A_iA_j forms a harmonic range with the derived pair proper to that line.

v. The 12 derived points constitute a Hessian $(12_4, 16_3)$ configuration. The array is (1) with A, B, C replaced by X, Y, Z respectively. Conversely, the first configuration can be obtained from the second in a similar manner.

vi. The vertices of the diagonal triangles of the six quadrangles are the same nine points, namely

$$
\begin{aligned}
&A_0A_3, A_1A_2 = P_{AX} = X_0X_1, X_2X_3, \quad && B_0B_3, B_1B_2 = P_{BX} = X_0X_2, X_1X_3,\\
&A_0A_2, A_1A_3 = P_{AY} = Y_0Y_1, Y_2Y_3, && B_0B_2, B_1B_3 = P_{BY} = Y_0Y_2, Y_1Y_3,\\
&A_0A_1, A_2A_3 = P_{AZ} = Z_0Z_1,\ Z_2Z_3\ ; && B_0B_1, B_2B_3 = P_{BZ} = Z_0Z_2,\ Z_1Z_3\ ;
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&C_0C_3,\ C_1C_2 = P_{CX} = X_0X_3,\ X_1X_2,\\
&C_0C_2,\ C_1C_3 = P_{CY} = Y_0Y_3,\ Y_1Y_2,\\
&C_0C_1,\ C_2C_3 = P_{CZ} = Z_0Z_3,\ Z_1Z_2.
\end{aligned}
$$

The diagonal triangles can be arranged by noting the subscripts. From the properties of "corresponding" points on a cubic (§ 1) it follows that

The two cubics C_6^3 and $C_6'^3$ determined by the conjugate configurations intersect at the vertices of the diagonal triangles of the six complete quadrangles.

3. **Some developments.** We wish to draw several corollaries from the properties which have been enumerated and to add some theorems.

i. The perspectivities given in (§ 2, iii) show that not only are the triangles $B_1B_2B_3$, $C_1C_2C_3$, $A_1A_2A_3$ centrally perspective in pairs from the collinear points A_0, B_0, C_0 and so axially perspective at X_1, Y_2, Z_3, but moreover the triangles $B_1C_1A_1$, $B_2C_2A_2$, $B_3C_3A_3$ are centrally perspective at X_1, Y_2, Z_3 and axially at A_0, B_0, C_0. Thus

For any three collinear points of a Hessian configuration the remaining nine not only form three triangles centrally perspective in pairs at the selected

points, and so axially perspective at three collinear points of an associated line, but moreover they form three triangles centrally perspective at the associated points and axially perspective at the initial points.

ii. In particular, the three points A, B, C form with the vertices of the diagonal triangles of the quadrangles $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ a $(12_4, 16_3)$ configuration. We have then the perspectivities

$$(4_1)\qquad \begin{vmatrix} P_{BX}P_{CX} \\ P_{BY}P_{CY} \\ P_{BZ}P_{CZ} \end{vmatrix} A \qquad \begin{vmatrix} P_{CX}P_{AX} \\ P_{CY}P_{AY} \\ P_{CZ}P_{AZ} \end{vmatrix} B \qquad \begin{vmatrix} P_{AX}P_{BX} \\ P_{AY}P_{BY} \\ P_{AZ}P_{CZ} \end{vmatrix} C$$

and there exist three collinear points X, Y, Z on $\mathcal{C}'^3_6$ such that we have the perspectivities

$$(4_2)\qquad \begin{vmatrix} P_{AY}P_{AZ} \\ P_{BY}P_{BZ} \\ P_{CY}P_{CZ} \end{vmatrix} X \qquad \begin{vmatrix} P_{AZ}P_{AY} \\ P_{BZ}P_{BX} \\ P_{CZ}P_{CX} \end{vmatrix} Y \qquad \begin{vmatrix} P_{AX}P_{AY} \\ P_{BX}P_{BY} \\ P_{CX}P_{CY} \end{vmatrix} Z.$$

iii. *For a given cubic of reference $\mathcal{C}^3_6$, each line $\mathcal{L}$ of the plane determines an associated cubic $\mathcal{C}'^3_6$, and an associated line $\mathcal{L}'$; and conversely, $\mathcal{L}'$ and $\mathcal{C}'^3_6$ determine in similar fashion the given cubic and line.*

The derived figures may be determined in either of two ways. If $\mathcal{L}$ cut $\mathcal{C}^3_6$ in A, B, C we can find the corresponding Hessian configuration, its conjugate configuration, and so the satellite line XYZ; or we can find the three sets of points cotangential with A, B, C, construct X, Y, Z from the intersections in (4_2), and then take the cubic on the 12 points before us.

The line $X_1Y_1Z_0$ cuts the sides of triangle $A_0A_1A_2$ as follows: A_1A_2 at X, A_2A_0 at Y_1 and A_0A_1 at Z_0 (cf. (2)); and on these sides lie X_0, Y_0, Z_1 respectively, forming harmonic ranges with the other three points. It is a theorem of elementary geometry that the six lines which join these six points to the vertices of the triangle meet in triples at four points Q_i, and the given triangle is the diagonal triangle of this quadrangle. (The bisectors of the angles of a triangle are an instance.) Next, in the quadrangle $A_2B_2Z_1C_1$: two sides pass through B_3, two through C_0 and one through A_3, while $A_3B_3C_0$ are collinear. Therefore A_2Z_0 cuts B_3C_0 in Q_3 harmonic to A_3 for B_3C_0. Similarly, we find that A_1Y_0 and A_0X_0 meet at Q_3. Finally, as Q_3 is harmonic to A_3 for B_3C_0 it is the point in which this line cuts the polar conic of A_3 with respect to $\mathcal{C}^3_6$. Accordingly,

The six points of the derived sets $\{X_i\}$, $\{Y_i\}$, $\{Z_i\}$ which lie in pairs on the sides of the (typical) triangle $A_0A_1A_2$ are such that the lines which

join them to the vertices A_i $(i=0, 1, 2)$ meet in triples at four points which lie on the polar conic of A_3 where that curve is cut by the four lines of collineation from A_3 to $\{B_i\}$, $\{C_i\}$. Triangle $A_0A_1A_2$ is the diagonal triangle of this inscribed four point.

From the definition of Q_3 it is evident that its polar line with respect to triangle $A_0A_1A_2$ is $X_1Y_1Z_0$. On the other hand, this line is the line associated with $A_3B_3C_0$ by the configuration. Hence

If a line cut a cubic C_6^3 in three points A, B, C, its associated line is the polar line, with respect to the triangle of the points cotangential to (say) A, for the point where the line cuts again the polar conic for A.

The polar line for a point P with respect to a triangle is dual to the quadratic transform of P under a properly chosen quadratic transformation of which that triangle is the base. Now, as a line rotates around A its intersection with the polar conic of A traces that conic, a quadratic transform of which is a curve of the fourth order and sixth class having nodes at the points cotangential to A. The dual envelope is a curve of the sixth order and fourth class doubly tangent to the sides of the triangle.

As a line rotates around a point A on a cubic C_6^3, its associated line envelops a curve of the sixth order and fourth class which has for double tangents the sides of the triangle of points cotangential to A.

Thus with each point of a cubic C_6^3 is associated a curve of the sixth order and fourth class.

The desmic configurations (§ 4–5)

4. Desmic tetrahedra. Let A_0, A_1, A_2, A_3 be the vertices of a tetrahedron $\{A\}$ and C_0 an arbitrary point. If C_0 be reflected harmonically with respect to the opposite edges of $\{A\}$, three new points C_1, C_2, C_3 are determined, which with C_0 form a tetrahedron $\{C\}$. Again, if C_0 be reflected harmonically with respect to each vertex of $\{A\}$ and its opposite face, four vertices of a third tetrahedron $\{B\}$ are determined. Three tetrahedra related in this manner have been named *desmic* by Stephanos (6*). They are the singular surfaces in a pencil of quartic surfaces. Desmic tetrahedra possess the important property that any two of them are in perspective from each of the vertices of the third. Thus the 12 vertices lie by triples on 16 lines with four lines through each vertex.

* This number refers to the papers listed at the end of the article.

As each vertex has all the properties that any one has, the whole configuration can be derived from any one of the tetrahedra and one extra vertex. Now when C_0 is reflected harmonically with respect to (say) A_0A_1 and A_2A_3 to obtain C_1, the line C_0C_1 cuts A_0A_1 in Z_1 and A_2A_3 in Z_2, and Z_1Z_2 are harmonic to C_0C_1. Thus on each edge of each tetrahedron we can determine two new points. Altogether there are only 12 of these and the tables of alignment are just those for the two Hessian configurations already considered. This second set of points gives three desmic tetrahedra, and conversely the first set can be determined from the second.

5. **Coördinates.** Choose $\{A\}$ as tetrahedron of reference and C_0 as unit point. The coördinates of the points in the two desmic systems are

$$
\begin{array}{lll}
A_0 = (1,0,0,0), & B_0 = (-1,1,1,1), & C_0 = (1,1,1,1), \\
A_1 = (0,1,0,0), & B_1 = (1,-1,1,1), & C_1 = (1,1,-1,-1), \\
A_2 = (0,0,1,0), & B_2 = (1,1,-1,1), & C_2 = (1,-1,1,-1), \\
A_3 = (0,0,0,1), & B_3 = (1,1,1,-1), & C_3 = (1,-1,-1,1); \\
X_0 = (0,1,1,0), & Y_0 = (1,0,1,0), & Z_0 = (1,-1,0,0), \\
X_1 = (0,1,-1,0), & Y_1 = (1,0,-1,0), & Z_1 = (1,1,0,0), \\
X_2 = (1,0,0,-1), & Y_2 = (0,1,0,-1), & Z_2 = (0,0,1,1), \\
X_3 = (1,0,0,1), & Y_3 = (0,1,0,1), & Z_3 = (0,0,1,-1).
\end{array} \tag{5}
$$

If a tetrahedron be regarded as a degenerate quartic surface of planes, the equations of the six surfaces are

$$
\begin{aligned}
&\{A\}: \ y_0\,y_1\,y_2\,y_3 = 0, \\
&\{B\}: \ [(y_0-y_1)^2-(y_2+y_3)^2]\cdot[(y_0+y_1)^2-(y_2-y_3)^2] = 0, \\
&\{C\}: \ [(y_0+y_1)^2-(y_2+y_3)^2]\cdot[(y_0-y_1)^2-(y_2-y_3)^2] = 0, \\
&\{X\}: \ (y_0^2-y_1^2)\cdot(y_2^2-y_3^2) = 0, \\
&\{Y\}: \ (y_2^2-y_0^2)\cdot(y_1^2-y_3^2) = 0, \\
&\{Z\}: \ (y_0^2-y_3^2)\cdot(y_1^2-y_2^2) = 0.
\end{aligned} \tag{6}
$$

In each system the sum of the left members is identically zero. By the transformation

$$y_0' = y_0 + y_1, \quad y_1' = y_0 - y_1, \quad y_2' = y_2 + y_3, \quad y_3' = y_2 - y_3$$

we can pass from one system to the other.

If $S = (x_0, x_1, x_2, x_3)$ be chosen as an arbitrary point in space, the quartic surface, in variables y,

$$
\begin{vmatrix}
(y_0^2-y_1^2)(y_2^2-y_3^2), & (y_0^2-y_2^2)(y_1^2-y_3^2) \\
(x_0^2-x_1^2)(x_2^2-x_3^2), & (x_0^2-x_2^2)(x_1^2-x_3^2)
\end{vmatrix} = 0 \tag{7}
$$

passes through (x) and has the 12 desmic points A, B, C as nodes. Each of the 16 lines such as $A_1B_2C_3$ lies entirely on this quartic .

The conjugate quartic surface

$$(8)\qquad \begin{vmatrix} y_0y_1y_2y_3, & [(y_0-y_1)^2-(y_2+y_3)^2]\cdot[(y_0+y_1)^2-(y_2-y_3)^2] \\ x_0x_1x_2x_3, & [(x_0-x_1)^2-(x_2+x_3)^2]\cdot[(x_0+x_1)^2-(x_2-x_3)^2] \end{vmatrix} = 0$$

has the desmic points X, Y, Z as nodes; and each line such as $X_1Y_2Z_3$ lies entirely on it.

Humbert (8* p. 356) observed that the generators of the nets of quadrics on the pairs of desmic tetrahedra of one set form a single cubic complex. Among the generators are bitangents of each surface $\mathcal{D}$, and every generator is a bitangent of some surface $\mathcal{D}$ on the desmic points. At a point S, the lines of the complex form a cubic cone which passes through the desmic points. The equation of the complex at (x) for the tetrahedra of $\mathcal{D}$ is

$$(9)\qquad p_{23}\cdot p_{31}\cdot p_{12}+p_{23}\cdot p_{03}\cdot p_{02}+p_{31}\cdot p_{03}\cdot p_{01}+p_{12}\cdot p_{02}\cdot p_{01}=0,$$

and for those of $\mathcal{D}'$ it is

$$(9')\qquad p_{01}\cdot p_{02}\cdot p_{03}+p_{01}\cdot p_{12}\cdot p_{13}+p_{02}\cdot p_{21}\cdot p_{23}+p_{30}\cdot p_{31}\cdot p_{32}=0.$$

In this same paper and in an earlier one (7*) Humbert studied the quartic curve which is a plane section of a single desmic surface and showed that there exists on such a curve a single infinity of sets of 16 points which lie by fours on 12 lines. The envelope of these lines is a curve of the third class. He remarked that the configuration of lines and envelope form the reciprocal of results obtained by Schroeter (5*) in his study of a single Hessian configuration.

We propose to show that the complete Hessian configuration *and its conjugate* may be obtained from a desmic system by projection and section; to develop some properties of the curves and of the surfaces; and to show some relations among them.

Cubic curves and desmic surfaces (§ 6–12)

6. **A correspondence between cubics and desmic systems.** It is evident that if the 24 desmic points be projected from an arbitrary point S upon a plane, we have two conjugate Hessian configurations such as already considered. For convenience take the plane of projection as $y_3=0$, and

* These numbers refer to the papers listed at the end of the article.

denote the projection of each point by the same letter. Then in the plane we have

$$
\begin{array}{lll}
A_0=(1,0,0), & B_0=(x_0+x_3, x_1-x_3, x_2-x_3), & C_0=(x_0-x_3, x_1-x_3, x_2-x_3),\\
A_1=(0,1,0), & B_1=(x_0-x_3, x_1+x_3, x_2-x_3), & C_1=(x_0+x_3, x_1+x_3, x_2-x_3),\\
A_2=(0,0,1), & B_2=(x_0-x_3, x_1-x_3, x_2+x_3), & C_2=(x_0+x_3, x_1-x_3, x_2+x_3),\\
A_3=(x_0, x_1, x_2), & B_3=(x_0+x_3, x_1+x_3, x_2+x_3), & C_3=(x_0-x_3, x_1+x_3, x_2+x_3),\\
X_0=(0,1,1), & Y_0=(1,0,1), & Z_0=(1,-1,0),\\
X_1=(0,1,-1), & Y_1=(1,0,-1), & Z_1=(1,1,0),\\
X_2=(x_0+x_3, x_1, x_2), & Y_2=(x_0, x_1+x_3, x_2), & Z_2=(x_0, x_1, x_2-x_3),\\
X_3=(x_0-x_3, x_1, x_2), & Y_3=(x_0, x_1-x_3, x_2), & Z_3=(x_0, x_1, x_2+x_3).
\end{array}
\tag{10}
$$

With this table of coördinates it is easy to verify that the equations of the two cubics are the following: for ABC

$$
\begin{aligned}
\mathcal{C}_6^3\colon\ & (x_0^2-x_3^2)y_1y_2(x_2y_1-x_1y_2)+(x_1^2-x_3^2)y_2y_0(x_0y_2-x_2y_0)\\
& +(x_2^2-x_3^2)y_0y_1(x_1y_0-x_0y_1)=0\,;
\end{aligned}
\tag{11}
$$

and for XYZ

$$
\begin{aligned}
\mathcal{C}'^3_6\colon\ & y_0(x_0y_2-x_2y_0)(x_1y_0-x_0y_1)+y_1(x_1y_0-x_0y_1)(x_2y_1-x_1y_2)\\
& +y_2(x_2y_1-x_1y_2)(x_0y_2-x_2y_0)+x_3^2\,y_0y_1y_2=0.
\end{aligned}
\tag{11'}
$$

From equations (10), (11), (11′) it is evident that if the desmic configurations be projected upon $y_3=0$ from $S'=(x_0,\ x_1,\ x_2,-x_3)$—a point which is the harmonic reflection of S with respect to A_3 and plane $y_3=0$—we obtain the same configurations with the quadrangles B and C interchanged. This remark enables us to construct a desmic configuration to correspond to a given Hessian configuration.

Let the configuration H be on a cubic $\mathcal{C}_6^3$ in a plane π. Choose an arbitrary center of projection S and a plane π', not on S, for face $A_0A_1A_2$ of tetrahedron $\{A\}$. Project H and $\mathcal{C}_6^3$ from S upon π' and denote the projected points by the same letters accented. Now, A_0', A_1', A_2' are A_0, A_1, A_2 for tetrahedron $\{A\}$. Take A_3 arbitrarily on SA_3'. Thus one tetrahedron is determined.

Take S' on SA_3' harmonic to S for A_3 and A_3'. From hypothesis $A_3'B_3'C_0'$ are collinear in π'. Therefore SB_3' and $S'C_0'$ meet in a point B_3, while SC_0' and $S'B_3'$ meet at a point C_0, and $A_3B_3C_0$ are collinear from the harmonic relations. B_0 and C_0 lie on the plane $S(A_0'B_0'C_0')$; they also lie in the plane $S'(C_3'B_3')$, and by hypothesis B_3', C_3' are collinear with $A_0'\equiv A_0$. Thus $A_0B_0C_0$ are collinear. Similarly, from the other

triads of collinear points in the plane we obtain collinear points in space, and so construct a desmic system which projects from S into the given Hessian configurations.

7. **Reflections with respect to the desmic tetrahedra.** Let $\mathfrak{A}_{23}$ denote the operation of reflecting a point S harmonically with respect to the opposite edges A_2A_3 and A_0A_1 of tetrahedron $\{A\}$, and let $\mathfrak{A}_i$ denote harmonic reflection with respect to vertex A_i and its opposite face. In terms of such a notation we have for the several tetrahedra the following operations:

$$(12)\quad \begin{array}{l} 1\sim(x_0,x_1,x_2,x_3),\ \mathfrak{A}_{23}(x_0,x_1,-x_2,-x_3),\ \mathfrak{A}_{13}(x_0,-x_1,x_2,-x_3),\ \mathfrak{A}_{12}(x_0,-x_1,-x_2,x_3),\\ \mathfrak{A}_0(-x_0,x_1,x_2,x_3),\ \mathfrak{A}_1(x_0,-x_1,x_2,x_3),\ \mathfrak{A}_2(x_0,x_1,-x_2,x_3),\ \mathfrak{A}_3(x_0,x_1,x_2,-x_3);\end{array}$$

$$(13)\quad \begin{array}{l} 1\sim(x_0,x_1,x_2,x_3),\ \mathfrak{C}_{23}(x_1,x_0,x_3,x_2),\ \mathfrak{C}_{13}(x_2,x_3,x_0,x_1),\ \mathfrak{C}_{12}(x_3,x_2,x_1,x_0),\\ \mathfrak{C}_0(-x_0+x_1+x_2+x_3):(x_0-x_1+x_2+x_3):(x_0+x_1-x_2+x_3):(x_0+x_1+x_2-x_3),\\ \mathfrak{C}_1(x_0-x_1+x_2+x_3):(-x_0+x_1+x_2+x_3):(x_0+x_1+x_2-x_3):(x_0+x_1-x_2+x_3),\\ \mathfrak{C}_2(x_0+x_1-x_2+x_3):(x_0+x_1+x_2-x_3):(-x_0+x_1+x_2+x_3):(x_0-x_1+x_2+x_3),\\ \mathfrak{C}_3(x_0+x_1+x_2-x_3):(x_0+x_1-x_2+x_3):(x_0-x_1+x_2+x_3):(-x_0+x_1+x_2+x_3);\end{array}$$

$$(14)\quad \begin{array}{l} 1\sim(x_0,x_1,x_2,x_3),\ \mathfrak{B}_{23}(x_1,x_0,-x_3,-x_2),\ \mathfrak{B}_{13}(x_2,-x_3,\ x_0,\ -x_1),\ \mathfrak{B}_{12}(x_3,-x_2,-x_1,x_0),\\ \mathfrak{B}_0(x_0+x_1+x_2+x_3):(x_0+x_1-x_2-x_3):(x_0-x_1+x_2-x_3):(x_0-x_1-x_2+x_3),\\ \mathfrak{B}_1(x_0+x_1-x_2-x_3):(x_0+x_1+x_2+x_3):(-x_0+x_1+x_2-x_3):(-x_0+x_1-x_2+x_3),\\ \mathfrak{B}_2(x_0-x_1+x_2-x_3):(-x_0+x_1+x_2-x_3):(x_0+x_1+x_2+x_3):(-x_0-x_1+x_2+x_3),\\ \mathfrak{B}_3(x_0-x_1-x_2+x_3):(-x_0+x_1-x_2+x_3):(-x_0-x_1+x_2+x_3):(x_0+x_1+x_2+x_3);\end{array}$$

$$(15)\quad \begin{array}{l} 1,\ \mathfrak{X}_{23}=\mathfrak{A}_{12},\ \mathfrak{X}_{13}=\mathfrak{B}_{12},\ \mathfrak{X}_{12}=\mathfrak{C}_{12},\\ \mathfrak{X}_0(x_0,-x_2,-x_1,x_3),\ \mathfrak{X}_1(x_0,x_2,x_1,x_3),\ \mathfrak{X}_2(x_3,x_1,x_2,x_0),\ \mathfrak{X}_3(x_3,-x_1,-x_2,x_0);\end{array}$$

$$(16)\quad \begin{array}{l} 1,\ \mathfrak{Y}_{23}=\mathfrak{A}_{13},\ \mathfrak{Y}_{13}=\mathfrak{B}_{13},\ \mathfrak{Y}_{12}=\mathfrak{C}_{13},\\ \mathfrak{Y}_0(-x_2,x_1,-x_0,x_3),\ \mathfrak{Y}_1(x_2,x_1,x_0,x_3),\ \mathfrak{Y}_2(x_0,x_3,x_2,x_1),\ \mathfrak{Y}_3(x_0,-x_3,-x_2,x_1);\end{array}$$

$$(17)\quad \begin{array}{l} 1,\ \mathfrak{Z}_{23}=\mathfrak{A}_{23},\ \mathfrak{Z}_{13}=\mathfrak{B}_{23},\ \mathfrak{Z}_{12}=\mathfrak{C}_{23},\\ \mathfrak{Z}_0(x_1,x_0,x_2,x_3),\ \mathfrak{Z}_1(-x_1,-x_0,x_2,x_3),\ \mathfrak{Z}_2(x_0,x_1,-x_3,-x_2),\ \mathfrak{Z}_3(x_0,x_1,x_3,x_2).\end{array}$$

The operations of (12) are of order 2 and form a group of order 8 in which 1, $\mathfrak{A}_{23}$, $\mathfrak{A}_{13}$, $\mathfrak{A}_{12}$ form a subgroup. It is easy to verify that all these points are on the desmic surface $\mathcal{D}$ determined by (x_0, x_1, x_2, x_3).

Every point of a desmic surface determines three desmic systems of points on the surface, each system containing one tetrahedron of nodes.

8. **A 16_6 configuration.** The three operations of reflecting S on the three pairs of opposite edges of a tetrahedron are each of period 2 and with identity form a group of order 4. It is evident that the operations $\mathfrak{A}$ and $\mathfrak{C}$ are commutative, wherefore their product is a group of order 16, which includes as a subgroup the reflections with respect to $\{B\}$. These 16 points form the well known 16_6 configuration of 16 points which lie in sixes on 16 planes which meet by sixes at the 16 points. The coördinates are

$$
\begin{array}{llll}
(x_0, x_1, x_2, x_3), & (x_1, x_0, x_3, x_2), & (x_2, x_3, x_0, x_1), & (x_3, x_2, x_1, x_0), \\
(x_0, x_1, -x_2, -x_3), & (x_1, x_0, -x_3, -x_2), & (x_2, x_3, -x_0, -x_1), & (x_3, x_2, -x_1, -x_0), \\
(x_0, -x_1, x_2, -x_3), & (x_1, -x_0, x_3, -x_2), & (x_2, -x_3, x_0, -x_1), & (x_3, -x_2, x_1, -x_0), \\
(x_0, -x_1, -x_2, x_3), & (x_1, -x_0, -x_3, x_2), & (x_2, -x_3, -x_0, x_1), & (x_3, -x_2, -x_1, x_0).
\end{array} \tag{18}
$$

It is evident that these 16 points all lie on both desmic surfaces which S determines. Moreover, the coördinates of the points are also the coördinates of the planes. For if the table (18) be arranged as follows as an "incidence table"

$$
\begin{array}{llll}
(x_0, x_1, x_2, x_3), & (x_3, x_2, -x_1, -x_0), & (x_1, -x_0, x_3, -x_2), & (x_2, -x_3, -x_0, x_1), \\
(x_2, x_3, -x_0, -x_1), & (x_1, x_0, x_3, x_2), & (x_3, -x_2, x_1, -x_0), & (x_0, -x_1, x_2, -x_3), \\
(x_3, -x_2, x_1, -x_0), & (x_0, -x_1, -x_2, x_3), & (x_2, x_3, x_0, x_1), & (x_1, x_0, -x_3, -x_2), \\
(x_1, -x_0, -x_3, x_2), & (x_2, -x_3, x_0, -x_1), & (x_0, x_1, -x_2, -x_3), & (x_3, x_2, x_1, x_0),
\end{array} \tag{19}
$$

we see that if any set be taken as the coördinates of a plane, then that plane contains the six points whose coördinates are in the row and column which cross at that set (9,* p. 7).

9. **The intersections of the cubics.** To find the points of the surfaces which give the vertices of the diagonal triangles of the Hessian configurations, we observe that P_{AX} is at the intersection of the planes SA_0A_1 and SA_2A_3; accordingly P_{AX} may be regarded as the projection of the harmonic reflection of S with respect to the opposite edges A_0A_1, A_2A_3 of tetrahedron $\{A\}$. These edges are also opposite in tetrahedron $\{X\}$, and we have found that the reflection of S lies on both desmic surfaces. Reflecting S with respect to the opposite edges in each tetrahedron of the systems we obtain the nine points desired.

---

* This number refers to the papers listed at the end of the article.

In (18) these points lie in the first row, the first column and in the principal diagonal. The projections give

$$
\begin{aligned}
&P_{AX} = (0, x_1, x_2), \quad P_{BX} = (x_0^2 - x_3^2,\ x_0x_1 + x_2x_3,\ x_0x_2 + x_1x_3),\\
&P_{AY} = (x_0, 0, x_2), \quad P_{BY} = (x_0x_1 + x_2x_3,\ x_1^2 - x_3^2,\ x_1x_2 + x_0x_3),\\
&P_{AZ} = (x_0, x_1, 0), \quad P_{BZ} = (x_0x_2 + x_1x_3,\ x_1x_2 + x_0x_3,\ x_2^2 - x_3^2),\\
&P_{CX} = (x_0^2 - x_3^2,\ x_0x_1 - x_2x_3,\ x_0x_2 - x_1x_3),\\
&P_{CY} = (x_0x_1 - x_2x_3,\ x_1^2 - x_3^2,\ x_1x_2 - x_0x_3),\\
&P_{CZ} = (x_0x_2 - x_1x_3,\ x_1x_2 - x_0x_3,\ x_2^2 - x_3^2).
\end{aligned} \tag{20}
$$

From these we find by the alignments (4)

$$
\begin{aligned}
A &= x_1x_2(x_0^2 - x_3^2),\ x_2x_0(x_1^2 - x_3^2),\ x_0x_1(x_2^2 - x_3^2),\\
B &= (x_0^2 - x_3^2)(x_1x_2 + x_0x_3),\ (x_1^2 - x_3^2)(x_0x_2 + x_1x_3),\ (x_2^2 - x_3^2)(x_0x_1 + x_2x_3),\\
C &= (x_0^2 - x_3^2)(x_1x_2 - x_0x_3),\ (x_1^2 - x_3^2)(x_0x_2 - x_1x_3),\ (x_2^2 - x_3^2)(x_0x_1 - x_2x_3);
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
X &= 2x_0(x_1^2 - x_2^2),\ x_1(x_0^2 + x_1^2 - x_2^2 - x_3^2),\ x_2(-x_0^2 + x_1^2 - x_2^2 + x_3^2),\\
Y &= x_0(-x_0^2 - x_1^2 + x_2^2 + x_3^2),\ 2x_1(x_2^2 - x_0^2),\ x_2(-x_0^2 + x_1^2 + x_2^2 - x_3^2),\\
Z &= x_0(x_0^2 - x_1^2 + x_2^2 - x_3^2),\ x_1(x_0^2 - x_1^2 - x_2^2 + x_3^2),\ 2x_2(x_0^2 - x_1^2).
\end{aligned} \tag{22}
$$

The equations of the satellite lines are

$$
\begin{aligned}
\mathcal{L}: (x_1^2 - x_3^2)(x_2^2 - x_3^2)(x_1^2 - x_2^2)x_0y_0 + (x_2^2 - x_3^2)(x_0^2 - x_3^2)(x_2^2 - x_0^2)x_1y_1\\
+ (x_0^2 - x_3^2)(x_1^2 - x_3^2)(x_0^2 - x_1^2)x_2y_2 = 0\ ;
\end{aligned} \tag{23}
$$

$$
\mathcal{L}': \begin{vmatrix}
y_0, & y_1, & y_2\\
x_0(-x_0^2 - x_1^2 + x_2^2 + x_3^2), & 2x_1(x_2^2 - x_0^2), & x_2(-x_0^2 + x_1^2 + x_2^2 - x_3^2)\\
x_0(x_0^2 - x_1^2 + x_2^2 - x_3^2), & x_1(x_0^2 - x_1^2 - x_2^2 + x_3^2), & 2x_0(x_0^2 - x_1^2)
\end{vmatrix} = 0. \tag{24}
$$

10. **Tangent planes.** The equation of the tangent plane at S to $\mathcal{D}$ is

$$
\begin{aligned}
&(x_1^2 - x_3^2)(x_2^2 - x_3^2)(x_1^2 - x_2^2)x_0y_0 + (x_2^2 - x_3^2)(x_0^2 - x_3^2)(x_2^2 - x_0^2)x_1y_1\\
&+ (x_0^2 - x_3^2)(x_1^2 - x_3^2)(x_0^2 - x_1^2)x_2y_2\\
&+ (x_0^2 - x_1^2)(x_1^2 - x_2^2)(x_2^2 - x_0^2)x_3y_3 = 0,
\end{aligned} \tag{25}
$$

which, evidently, cuts the plane $y_3 = 0$ in the line $\mathcal{L}$. Accordingly

If two conjugate desmic systems be projected from a point S upon a plane into two conjugate Hessian configurations, the tangent planes at S to the conjugate desmic surfaces cut the plane of projection in the satellite lines of the configurations.

The tangent plane at a point of a desmic surface can be constructed linearly as follows:

Project the system of nodes from the point upon a convenient plane and obtain the nine points P_{AX}; use the alignments (4) to find A, B, C. The plane $SABC$ is the tangent plane.

The tangent plane to $\mathcal{D}$ at S cuts that surface in a quartic curve which has a singular point at S. From S there are six tangents to the quartic and so six bitangents to the desmic surface. Three of these can be obtained directly. The points A, B, C on $y_3=0$ project back upon $\mathcal{D}$ into

$$
\begin{aligned}
A &= (x_1x_2x_3,\ x_0x_2x_3,\ x_0x_1x_3,\ x_0x_1x_2)\ ;\\
B &= 2x_1x_2x_3+x_0(-x_0^2+x_1^2+x_2^2+x_3^2),\ 2x_0x_2x_3+x_1(x_0^2-x_1^2+x_2^2+x_3^2),\\
&\qquad 2x_0x_1x_3+x_2(x_0^2+x_1^2-x_2^2+x_3^2),\ 2x_0x_1x_2+x_3(x_0^2+x_1^2+x_2^2-x_3^2)\ ;\\
C &= 2x_1x_2x_3-x_0(-x_0^2+x_1^2+x_2^2+x_3^2),\ 2x_0x_2x_3-x_1(x_0^2-x_1^2+x_2^2+x_3^2),\\
&\qquad 2x_0x_1x_3-x_2(x_0^2+x_1^2-x_2^2+x_3^2),\ 2x_0x_1x_2-x_3(x_0^2+x_1^2+x_2^2-x_3^2).
\end{aligned}
\tag{26}
$$

A rather tedious calculation shows that the line SA cuts $\mathcal{D}$ in two coincident points at A, and similarly for the lines SB and SC. Moreover, we observe that the points A, B, C on the surface are collinear.

On $\mathcal{D}'$ we have the points

$$
\begin{aligned}
X &= x_0(x_1^2-x_2^2):x_1(x_0^2-x_3^2):-x_2(x_0^2-x_3^2):-x_3(x_1^2-x_2^2)\ ;\\
Y &= x_0(x_1^2-x_3^2):x_1(x_0^2-x_2^2):-x_2(x_1^2-x_3^2):-x_3(x_0^2-x_2^2)\ ;\\
Z &= x_0(x_2^2-x_3^2):-x_1(x_2^2-x_3^2):x_2(x_0^2-x_1^2):-x_3(x_0^2-x_1^2).
\end{aligned}
\tag{27}
$$

We recall (§ 6) that there is a cubic complex for each space system of desmic points and among the lines of the complex are bitangents of $\mathcal{D}$. At a point S the lines of the complex form a cubic cone which passes through the desmic points. *This cubic cone at S for $\mathcal{D}$ cuts $y_3=0$ in the cubic curve $\mathcal{C}_6^3$, while the cone for $\mathcal{D}'$ cuts $y_3=0$ in $\mathcal{C}_6'^3$.* This may be verified at once with $S=(x)$ in the equations (9) and (9′). The bitangents SA, SB, SC are the section of the first cone by the tangent plane to $\mathcal{D}$. It might be surmised that the three other bitangents from S to $\mathcal{D}$ are the lines in which the tangent plane to $\mathcal{D}$ at S is cut by the *cone for $\mathcal{D}'$*. We find the intersections of the line $\mathcal{L}$ with the cubic $\mathcal{C}_6'^3$ and test the lines which join these intersections to S for bitangency. Unfortunately, they are not bitangents.

11. **A Poncelet figure and the cubics.** Let us return to the incidence table (19) of the 16_6 configuration. The polar plane of $S(x_0, x_1, x_2, x_3)$ with respect to the quadric

$$
\mathcal{Q}:\quad x_0^2+x_1^2+x_2^2+x_3^2=0 \tag{28}
$$

is $\sigma(x_0, x_1, x_2, x_3)$ and from the table we see that it contains the six points

$$
\begin{aligned}
&L\ (x_3, x_2, -x_1, -x_0), && L'\ (x_3, -x_2, x_1, -x_0),\\
&M\ (x_2, -x_3, -x_0, x_1), && M'\ (x_2, x_3, -x_0, -x_1),\\
&N\ (x_1, -x_0, x_3, -x_2), && N'\ (x_1, -x_0, -x_3, x_2).
\end{aligned} \tag{29}
$$

The polar plane for L is $\lambda(x_3, x_2, -x_1, -x_0)$ and the table shows that it contains the points S, M, N. Thus $SLMN$, or $\sigma\lambda\mu\nu$, is a self-polar tetrahedron with respect to $\mathfrak{Q}$. Similar results follow for the other three points. Again, the table shows that each of the remaining nine points of the configuration lies on two of the six planes.

It is known that the six points lie on a conic and the six planes envelop a quadric cone at S (9,* p. 12). On projection from S and section by an arbitrary plane, say $y_3=0$, we obtain a Poncelet figure of two triangles LMN and $L'M'N'$ inscribed in one conic and circumscribed to another. The six sides of the triangles cut each other in nine other points, three on each side, and these are the nine points P_{ij} of intersection of the cubics $\mathcal{C}_6^3$ and $\mathcal{C}'^3_6$.

The six lines as tangents to a conic give sixty hexagons and so sixty Brianchon points. But these six lines are at present being considered in two sets of three and we are not to take the vertices of these two triangles as the intersections of opposite sides of a hexagon. When one triangle of three lines is chosen, its sides may be paired with those of the other in six ways; so we have six hexagons and six Brianchon points. The Brianchon points lie by threes on twenty Steiner lines (3*). In the present instance the six points always lie on two such lines which are harmonic with respect to the conic. The points and lines are our ABC and XYZ, the pairings being made as follows. Consider the interior of the array

	l	m	n
l'	P_{AX}	P_{BZ}	P_{CY}
m'	P_{CZ}	P_{AY}	P_{BX}
n'	P_{BY}	P_{CX}	P_{AZ}

as a determinant of the third order, and pick out the triples that correspond to positive terms. These belong to ABC; those for the negative terms to XYZ.

We have the following theorems:

Given a Poncelet figure of two triangles circumscribed to a conic, the sides of one triangle cut the sides of the other in nine points which may be grouped

* These numbers refer to the papers listed at the end of the article.

in two ways in three triangles perspective in pairs from three Brianchon points on one Steiner line and axially perspective on the conjugate Steiner line. For the second grouping the lines are the same with their rôles exchanged. The two overlapping sets of twelve points form Hessian configurations on two cubics which intersect in the nine common points.

Conversely, *if three collinear points be deleted from a Hessian configuration on a cubic, the remaining points lie on six lines which form two triangles circumscribed to one conic and inscribed in another.*

Five of the lines determine a conic; let n'' be the second tangent from P_{AZ} to this conic. Then the properties of multiple perspectivity and the converse of Brianchon's theorem enable us to show that n'' coincides with n'.

12. **Theorems on the 16_6 configuration.** The points S, $\mathfrak{A}_{23}$, $\mathfrak{A}_{13}$, $\mathfrak{A}_{12}$, are conjugate with respect to the tetrahedron $\{A\}$ (§ 7). Their polar planes with respect to the quadric $\mathfrak{Q}$ (28) are

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0,$$
$$x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 = 0,$$
$$x_0y_0 - x_1y_1 + x_2y_2 - x_3y_3 = 0,$$
$$x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3 = 0.$$

The last three of these intersect at the point of contact A (26) on $\mathcal{D}$ of the bitangent SA. These three planes belong to the 16_6 configuration. Moreover, if a new point be determined in this manner with respect to $\{A\}$ for each point of the configuration, the 16 points form the 16_6 configuration generated from A with respect to the fundamental tetrahedra. Three such derived systems can be found on $\mathcal{D}$, one for each of the basal tetrahedra, and three on $\mathcal{D}'$.

The planes of a 16_6 configuration meet in threes in the points of six new 16_6 configurations which lie in two triads on the two conjugate desmic surfaces whose curve of intersection passes through the 16 given points. The points of a triad are collinear on 16 lines.

If we recall that we were led to the results of § 11 from a 16_6 configuration, we can say

When a section is taken of the six configuration planes on a point of a 16_6 configuration, the six lines give two Poncelet triangles and the nine intersections of their sides are cotangential in two sets of triples on two cubic curves.

NINE POINT CHECKERBOARDS ON A CUBIC (§§ 13–15)

13. **Darboux's coördinates.** For an analytic treatment of the problem of the Poncelet triangles it is convenient to use a system of coördinates introduced by Darboux. The conic $\mathcal{C}$ to which the triangles are to be tangent is taken as fundamental. Imagine that a system of ordinary point and line homogeneous coördinates has been taken such that the point equation of $\mathcal{C}$ is

$$y^2 = 4xz.$$

Then its line equation is

$$\eta^2 = \xi\zeta.$$

The coördinates of the points of $\mathcal{C}$ can be given in terms of a parameter t as

(i) $$x = t^2, \quad y = 2t, \quad z = 1,$$

whence the coördinates of the tangent at t are

(ii) $$\xi = 1, \quad \eta = -t, \quad \zeta = t.$$

The tangents at two points t_1 and t_2 meet at the pole of the line t_1, t_2:

(iii) $$x = t_1t_2, \quad y = t_1 + t_2, \quad z = 1;$$

and conversely, each point of the plane determines two points t_1, t_2 which are the intersections of $\mathcal{C}$ with its polar. Again two points t_1, t_2 are joined by the line of coördinates

(iv) $$\xi = 1, \quad \eta = -\tfrac{1}{2}(t_1 + t_2), \quad \zeta = t_1t_2.$$

Accordingly

(v) $$\xi t_1t_2 + \eta(t_1 + t_2) + \zeta = 0$$

is the cartesian equation of the point t_1, t_2; and

(vi) $$x - \tfrac{1}{2}y(t_1 + t_2) + zt_1t_2 = 0$$

is the cartesian equation of the line t_1, t_2.

More generally, the quadratic equation in t

$$at^2 + 2bt + c = 0$$

has two roots t_1, t_2 such that $t_1+t_2 = -2b/a$, $t_1\,t_2 = c/a$; and can be regarded as defining either the point

$$x : y : z = c : -2b : a$$

or the line

$$\xi : \eta : \zeta = a : b : c.$$

A curve of order n in x, y, z is by (iii) a form of order n in each of the variables t_1, t_2 and symmetric in the two. Conversely, such a symmetric form can be expressed as a form of order n in x, y, z. The $2n$ values, t, obtained when $t_1 = t_2$, are the $2n$ points where the curve cuts the conic. For a line $(\alpha t_1)\ (\beta t_2) = (\alpha t_2)\ (\beta t_1)$; and for a line tangent at $t = \tau_1$, $|\tau_1 t_1| \cdot |\tau_1 t_2| = 0$.

14. **The triangles circumscribed to a conic.** Let us consider the figure of two triangles circumscribed to a conic, the parameters of the points of contact being τ_1, τ_2, τ_3 for the one, and τ_4, τ_5, τ_6 for the other. The equations of the sides are

$$(t_1 - \tau_i)\ (t_2 - \tau_i) = 0 \qquad (i = 1, 2, \cdots, 6)$$

or more briefly

$$T_i^2 = 0.$$

The equation of the first triangle regarded as a degenerate cubic is

$$T_1^2\ T_2^2\ T_3^2 = 0,$$

and of the second

$$T_4^2\ T_5^2\ T_6^2 = 0.$$

Accordingly, the equation of the pencil of cubics through the nine points of intersection of the sides of the first triangle with those of the second is

$$T_1^2 T_2^2 T_3^2 + k\ T_4^2 T_5^2 T_6^2 = 0.$$

This equation gives a cubic curve since the factor T_i^2 gives a form linear in x and y.

Denote the intersection of tangents τ_1 and τ_4 by P_{AX}, and the other points as indicated in the array

	1	2	3
4	P_{AX}	P_{BZ}	P_{CY}
5	P_{CZ}	P_{AY}	P_{BX}
6	P_{BY}	P_{CX}	P_{AZ}

Think of the elements in the array as the elements of a determinant and pick out the positive terms, obtaining three triangles whose vertices (denoted by the indices) are

$$(1, 4)\ (2, 5)\ (3, 6) \quad (1, 5)\ (2, 6)\ (3, 4) \quad (2, 4)\ (3, 5)\ (1, 6).$$

If the first triple be points of a cotangential set on the cubic, their tangents meet on the curve. The tangent at a point on a curve is the polar line of the point. Write the cubic at length as

$$(t_1 - \tau_1)\ (t_2 - \tau_1)\ (t_1 - \tau_2)\ (t_2 - \tau_2)\ (t_1 - \tau_3)\ (t_2 - \tau_3)$$
$$+ k(t_1 - \tau_4)\ (t_2 - \tau_4)\ (t_1 - \tau_5)\ (t_2 - \tau_5)\ (t_1 - \tau_6)\ (t_2 - \tau_6) = 0$$

and polarize for each point. The tangent at (τ_1, τ_4) is

$$(t_1 - \tau_1)(t_2 - \tau_1)\overline{12}\cdot\overline{13}\cdot\overline{42}\cdot\overline{43} + k(t_1 - \tau_4)(t_2 - \tau_4)\overline{15}\cdot\overline{16}\cdot\overline{45}\cdot\overline{46} = 0,$$

where $\overline{ij}$ is an abbreviation for $(\tau_i - \tau_j)$. The tangent at (τ_2, τ_5) is

$$(t_1 - \tau_2)(t_2 - \tau_2)\overline{21}\cdot\overline{23}\cdot\overline{51}\cdot\overline{53} + k(t_1 - \tau_5)(t_2 - \tau_5)\overline{24}\cdot\overline{26}\cdot\overline{54}\cdot\overline{56} = 0;$$

and for (τ_3, τ_6)

$$(t_1 - \tau_3)(t_2 - \tau_3)\overline{31}\cdot\overline{32}\cdot\overline{61}\cdot\overline{62} + k(t_1 - \tau_6)(t_2 - \tau_6)\overline{34}\cdot\overline{35}\cdot\overline{64}\cdot\overline{65} = 0.$$

If these three tangents meet in one point on the cubic, k must be such that they and the cubic are satisfied simultaneously. Transpose the k term in the equation of each tangent, and multiply the results obtaining

$$T_1^2 T_2^2 T_3^2 = -k^3\left(\frac{\overline{45}\cdot\overline{46}\cdot\overline{56}}{\overline{12}\cdot\overline{13}\cdot\overline{23}}\right)T_4^2 T_5^2 T_6^2.$$

In the equation of the cubic transpose the k term and divide the equation above; then

$$k = \pm\frac{\overline{12}\cdot\overline{13}\cdot\overline{23}}{\overline{45}\cdot\overline{46}\cdot\overline{56}}.$$

It remains to determine the sign of k. The equations of the three tangents become with this ambiguous value

$$T_1^2\overline{23}\cdot\overline{34}\cdot\overline{56} \pm T_4^2\overline{15}\cdot\overline{16}\cdot\overline{23} = 0,$$
$$T_2^2\overline{15}\cdot\overline{35}\cdot\overline{46} \pm T_5^2\overline{13}\cdot\overline{24}\cdot\overline{26} = 0,$$
$$T_3^2\overline{16}\cdot\overline{26}\cdot\overline{45} \pm T_6^2\overline{12}\cdot\overline{34}\cdot\overline{35} = 0.$$

If the tangents be concurrent these three equations must be linearly dependent. Impose this condition and reduce; then $k = -1$.

If we group the nine points into triples corresponding to the negative terms of the determinant form and impose similar conditions, as before we obtain the same quadratic for k, but when the sign is determined we have $k = +1$.

Thus we have proved the first part of the theorem of § 11.

On the first cubic, the common tangential point for P_{AX}, P_{AY}, P_{AZ} is $\bar{A}$; for P_{BX}, P_{BY}, P_{BZ} is $\bar{B}$; and for P_{CX}, P_{CY}, P_{CZ} is $\bar{C}$. The equation of the satellite line $\overline{ABC}$ is

$$\left\{\frac{T_1^2}{\overline{23}\,c_1} + \frac{T_2^2}{\overline{31}c_2} + \frac{T_3^2}{\overline{12}c_3}\right\} - \left\{\frac{T_4^2}{\overline{56}\,d_4} + \frac{T_5^2}{\overline{64}\,d_5} + \frac{T_6^2}{\overline{45}\,d_6}\right\} = 0,$$

where

$$c_i = \overline{i4} \cdot \overline{i5} \cdot \overline{i6}, \qquad d_m = \overline{m1} \cdot \overline{m2} \cdot \overline{m3}.$$

In general then if the indices $1, 2, \cdots, 6$ be grouped into two sets ijk and lmn, the two triangles of tangents to the fundamental conic indicated by these sets meet in nine points which form three cotangential sets in two Hessian $(12_4, 16_3)$ configurations, one on each of two cubics

$$T_i^2 T_j^2 T_k^2 (lmn) \pm T_l^2 T_m^2 T_n^2 (ijk) = 0$$

(where (lmn) denotes $\overline{lm} \cdot \overline{mn} \cdot \overline{nl}$), with the respective satellite lines

$$\left\{ \frac{T_i^2}{\overline{jk} c_i} + \frac{T_j^2}{\overline{ki} c_j} + \frac{T_k^2}{\overline{ij} c_k} \right\} \pm \left\{ \frac{T_l^2}{\overline{mn} d_l} + \frac{T_m^2}{\overline{nl} d_m} + \frac{T_n^2}{\overline{lm} d_n} \right\} = 0,$$

$$c_i = \overline{il} \cdot \overline{im} \cdot \overline{in}, \qquad d_m = \overline{im} \cdot \overline{jm} \cdot \overline{km}.$$

Conversely, if three collinear points be deleted from a Hessian configuration on a cubic, the remaining points lie on six lines which form two triangles circumscribed to one conic and inscribed in another.

Let the curve be referred to an elliptic parameter, whose values for four arbitrary points A, B, A', B', respectively, are congruent ($\equiv$)mod ω, ω' to u, v, w, t. Line AB cuts the curve in C with parameter $-(u+v)$; $A'B'$ in C' with $-(w+t)$; while AA' meets the curve in A'' with $-(u+w)$, and BB' in B'' with $-(v+t)$. Now the lines CC' and $A''B''$ meet on the cubic at C'' of parameter $(u+v+w+t)$. As the initial points can be chosen in ∞^4 ways there are that many sets of nine points of the cubic which lie on two triangles. These triangles, however, are not necessarily tangent to a conic.

If we add the condition that A, B', C'' be cotangential, then

$$-2u \equiv -2t \equiv -2(u+v+w+t) \qquad (\text{mod } (\omega, \omega')) \tag{a}$$

whence

$$-2v \equiv 2(w+t) \equiv 2(u+w)$$

and

$$-2w \equiv 2(v+t) \equiv 2(u+v).$$

The first of these congruences shows that A', B'', C are cotangential and the second that A'', B, C' are. Again if A'', B', C be cotangential so are A', B, C'' and A, B'', C'; for

$$2(u+w) \equiv -2t \equiv 2(u+v), \qquad (\text{mod } (\omega, \omega')) \tag{b}$$

implies

$$-2w \equiv -2v \equiv -2(u+v+w+t),$$

and

$$-2u \equiv 2(v+t) \equiv 2(w+t).$$

Moreover, the conditions (a) and (b) cannot both be true for the same cubic. These are the only ways that triangles can be selected from the nine points without having one of the six lines for a side, and we have proved that if one triangle has cotangential vertices so do the other triangles of its set. And when the points u and v are given the rest are determined. The projective specification of the cubic involves one absolute constant, and two are required for the two points, a total of three for the configuration.

On the other hand, we have proved that if two triangles be circumscribed to a conic, the nine intersections of the sides form such a configuration on either of two cubics, according to the grouping of the cotangential triples. To specify the conic and one triangle uses the eight constants for the determination of a projective system, and the second triangle requires just three absolute constants. Thus the two configurations are equivalent projectively.

15. **Checkerboards on the cubic.** As remarked in § 14, when four points are chosen on a cubic $\mathcal{C}$ two triples of lines can be determined such that the nine intersections of one triple with the other lie on $\mathcal{C}$. There are ∞^4 such sets on $\mathcal{C}$ and the lines are not necessarily tangent to a conic. This is a particular case of Coble's (10,* p. 10–12) "checkerboard configuration" consisting of the n^2 points which are the intersections of one set of n lines with another set of n lines. He has shown that in the present instance the configuration is poristic when the lines are tangent to a conic, that is, when one such configuration exists for a conic and cubic there is an infinity for the two curves. Now each cubic $\mathcal{C}_k$ of the pencil

$$T_1^2T_2^2T_3^2 - kT_4^2T_5^2T_6^2 = 0$$

has one checkerboard with respect to the base conic which we have used. (It is to be observed, that for only two cubics of the pencil, $k = \pm 1$, does the initial checkerboard have the additional property of falling into cotangential sets.) We propose to determine the system of checkerboards on a cubic $\mathcal{C}_k$.

An arbitrary tangent, parameter t_1, to the conic cuts $\mathcal{C}_k$ in three points and these at once determine the second triangle of tangents. Moreover, any one of these tangents, parameter t_2, cuts $\mathcal{C}_k$ in two other points which

* This number refers to the papers listed at the end of the article.

at once determine the other sides of the first triangle. We desire a function $f(t_1, t_2)=0$ which relates the two parameters so that when one point of contact is chosen the other five can be found.

The first triangle can be regarded as determined by a binary cubic form in t_1; let two of its positions be given by $(\alpha t_1)^3=0$ and $(\beta t_1)^3=0$. The pencil of all its positions is

$$(\alpha t_1)^3 + \lambda(\beta t_1)^3 = 0. \tag{i}$$

Similarly, the pencil for the second triangle may be written

$$(\gamma t_2)^3 + \mu(\delta t_2)^3 = 0. \tag{ii}$$

Now, in (i) each value of t_1 determines a corresponding λ and thereupon the two other values of t_1 which imply the same λ. Conversely, each λ determines a triangle. Similar relations are true in (ii) for μ and t_2. Next, each value of t_1 in (i) must give the triangle (ii); that is, λ must determine μ uniquely; and conversely. Moreover, the two triangles are symmetrically and involutorily related, for when the second triangle moves to a position held by the first, the latter must take the position left by the former. Therefore we can write

$$\begin{aligned} (\alpha t_1)^3 + \lambda(\beta t_1)^3 &= 0, \\ (\alpha t_2)^3 - \lambda(\beta t_2)^3 &= 0, \end{aligned} \tag{iii}$$

where $(\alpha t)^3=0$ and $(\beta t)^3=0$ are the two positions where the triangles of a pair coincide. As these relations are to hold for all values of λ the function f may be given the form

$$\begin{vmatrix} (\alpha t_1)^3, & (\beta t_1)^3 \\ (\alpha t_2)^3, & -(\beta t_2)^3 \end{vmatrix} = 0. \tag{iv}$$

Thus the general form of the function f has been determined, but to be complete it must contain the hypothesis that the checkerboard actually exists for the particular cubic

$$T_1^2T_2^2T_3^2 - kT_4^2T_5^2T_6^2 = 0.$$

In other words, we must determine the α and β functions, i. e., the two degenerate checkerboards, in terms of the given τ_i. It is evident that the checkerboard degenerates when $t_1=t_2$; that is, the points to be given by α and β are those at which the conic and cubic intersect. Accordingly, we have

$$\begin{aligned} (\alpha t)^3 &= \rho[(t-\tau_1)(t-\tau_2)(t-\tau_3) + \sqrt{k}(t-\tau_4)(t-\tau_5)(t-\tau_6)], \\ (\beta t)^3 &= \rho[(t-\tau_1)(t-\tau_2)(t-\tau_3) - \sqrt{k}(t-\tau_4)(t-\tau_5)(t-\tau_6)]. \end{aligned}$$

These values in (iv) give the desired $f(t_1^3, t_2^3) = 0$.

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EXPANSION PROBLEMS IN CONNECTION WITH HOMOGENEOUS LINEAR q-DIFFERENCE EQUATIONS*

BY

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I. Introduction

1. **Statement of the problem.** This paper has for its purpose the development of expansion theories for the general homogeneous linear q-difference equation with analytic coefficients analogous to those of Neumann and Gegenbauer for Bessel's equation.†

In Part II is treated the expansion of an arbitrary analytic function of a single variable in series analogous to those of Maclaurin and Laurent. In Part III expansions of functions of several variables are considered.

II. Expansion of analytic functions of a single variable

2. **Solutions of the q-difference equation.** In the equation

$$A_0(x)u_n(q^k x) + A_1(x)u_n(q^{k-1}x) + \cdots + A_{k-1}(x)u_n(qx) + (\lambda_n + A_k(x))u_n(x) = 0 \tag{1}$$

we shall suppose that q is a constant greater than unity in absolute value, that k is a fixed positive integer, that n is a non-negative integral parameter, and that the coefficients $A_0(x), A_1(x), \cdots, A_k(x)$ are analytic functions of x at $x=0$, $A_0(0)$ being different from zero. Let R be the absolute value of the singularity nearest $x=0$ of any function of the set $1/A_0(x), A_0(x), A_1(x), \cdots, A_k(x)$ and for $|x|<R$ let $A_i(x)$ have the expansion

$$A_i(x) = \sum_{j=0}^{\infty} a_{ij}x^j \qquad (i = 0,1,2, \cdots, k). \tag{2}$$

The parameter λ_n shall be so determined that equation (1) has a solution of the form

$$u_n(x) = x^{n+\nu}\left(1 + \sum_{s=1}^{\infty} c_{ns}x^s\right), \tag{3}$$

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† See Watson, *Theory of Bessel Functions*, London, 1922, where these results are summarized.

where ν is any constant. To see that this is possible, substitute this series for $u_n(x)$ in (1), and equate to zero the coefficients of the various powers of x. This formal procedure gives the results

$$a_{00}q^{k(n+\nu)} + a_{10}q^{(k-1)(n+\nu)} + \cdots + a_{k0} + \lambda_n = 0 \tag{4}$$

and

$$c_{ns}(\lambda_n - \lambda_{n+s}) = \sum_{i=0}^{s-1} c_{ni}S_{n,s-i}^{(i)} \qquad (s = 1,2,3, \cdots), \tag{5}$$

where

$$S_{nr}^{(i)} = -\sum_{j=0}^{k} a_{jr}q^{(k-j)(n+\nu+i)}$$

and c_{n0} is to be taken equal to unity.

Equation (4) will be taken as the definition of λ_n.* By (5) the coefficients c_{ns} are uniquely determined, provided that the relation

$$\lambda_n - \lambda_{n+s} \neq 0 \tag{6}$$

obtains for all non-negative integers n and all positive integers s. By forming the difference $\lambda_n - \lambda_{n+s}$, it is easy to see that as either n or s becomes infinite or as both become infinite simultaneously the absolute value of this difference becomes infinite. Consequently relation (6) holds for all except at most a finite number of pairs of values of n and s. It will be supposed that the quantities $q, k, \nu, a_{00}, a_{10}, \cdots, a_{k0}$ are such that (6) holds for this finite number also.

It will now be shown that, with the c_{ns} determined by (5), the series for $u_n(x)$ in (3) is convergent† for $|x| < |q|^k R$ and represents a solution of (1) for $|x| < R$. Since the expansions (2) are valid for $|x| < R$, a constant a exists such that

$$|a_{ij}| < a^j$$

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*If instead of equation (1) we write

$$A_0(x)u_n(q^k x) + A_1(x)u_n(q^{k-1}x) + \cdots + A_k(x)u_n(x) + \lambda_n u_n(q^h x) = 0,$$

where h is a non-negative integer not greater than k, equation (4) becomes

$$a_{00}q^{k(n+\nu)} + a_{10}q^{(k-1)(n+\nu)} + \cdots + a_{k0} + \lambda_n q^{h(n+\nu)} = 0,$$

while equation (5) becomes

$$c_{ns}q^{h(n+s+\nu)}(\lambda_n - \lambda_{n+s}) = \sum_{i=0}^{s-1} c_{ni}S_{n,s-i}^{(i)}.$$

Using instead of (8) the relation

$$|q^{-(k-h)(s+n)}(\lambda_n - \lambda_{n+s})| > 1/M,$$

the convergence of the series corresponding to (3) follows just as it does when $h = 0$.

† If the real part of $n+\nu$ is negative the terms of the series are not defined for $x = 0$. The series multiplied by $x^{-\nu}$, however, is convergent in this case.

for every i and j. Consequently there is a constant A such that

$$| S^{(s-r)}q^{-k(s+n)} | < A^r \tag{7}$$

for every s, n, and r. Also $q^{-k(s+n)}(\lambda_n-\lambda_{n+s})$ is bounded away from zero as either n or s becomes infinite or as both become infinite simultaneously, so that by virtue of (6) we may infer the existence of an $M>1$ such that

$$| q^{-k(s+n)}(\lambda_n - \lambda_{n+s}) | > 1/M \tag{8}$$

for every s and n. Using the bounds (7) and (8) we have by induction from (5) that

$$| c_{ns} | < 2^{s-1}M^sA^s,$$

or, putting $K=2MA$,

$$| c_{ns} | < K^s \tag{9}$$

for every s and n. Hence the series (3) is absolutely convergent for $|x|>K^{-1}$. It represents a solution of (1) for $|x|<|q|^{-k}K^{-1}$ since then the operations performed in the calculation of the coefficients c_{ns} are justifiable. From equation (1) one sees that the singularity nearest the point $x=0$ of any solution of (1) which is analytic at $x=0$ is at least a distance R from $x=0$. We may then conclude that the series

$$x^{n+\nu}\left(1 + \sum_{s=1}^{\infty} c_{ns}x^s\right)$$

is absolutely convergent* for $|x|<|q|^k R$ and represents a solution of (1) for $|x|<R$.†

3. **Expansion of x^m.** Let us consider the possibility of expanding x^m in the form

$$x^m = \sum_{n=m}^{\infty} b_{mn}U_n(x), \tag{10}$$

where we have written $U_n(x)$ for $x^{-\nu}u_n(x)$. If we substitute for $U_n(x)$ its

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* See the second footnote on p. 524.

† The existence of this solution might have been inferred from the general existence theorem given by R. D. Carmichael in the American Journal of Mathematics, vol. 34, p. 159, where, however, a different method is used. The fact that the present method may be employed is stated in this paper. The details are set down here in order to obtain the inequality (9) which will be used later.

power series expansion, formally interchange the orders of summation, and equate coefficients of like powers of x, we have

$$(11)\quad \begin{aligned} &b_{mm} = 1, \\ &b_{m,m+s} + b_{m,m+s-1}c_{m+s-1,1} + b_{m,m+s-2}c_{m+s-2,2} + \cdots + b_{mm}c_{ms} = 0, \quad s > 0. \end{aligned}$$

Using the inequality (9) we have by induction from (11) that

$$(12)\quad |b_{m,m+s}| < (2K)^s.$$

The repeated series

$$\sum_{n=m}^{\infty} b_{mn}\left(x^n + \sum_{s=1}^{\infty} c_{ns}x^{n+s} \right)$$

is therefore absolutely convergent for $|x|<1/2K$. Hence, if $|x|<1/2K$, the rearrangement of this series which was made in obtaining formulas (11) is justified and the expansion (10), (11) is valid.

4. **Expansion of $1/(t-x)$.** The expansion

$$(13)\quad 1/(t-x) = \sum_{m=0}^{\infty} x^m/t^{m+1}$$

is valid if only $|x|<|t|$. For $|x|<1/2K$ we may infer from (9) and (12) the existence of M_1 and M_2 independent of n such that

$$\left| 1 + \sum_{s=1}^{\infty} c_{ns}x^s \right| < M_1$$

and

$$\left| \sum_{s=0}^{\infty} b_{n,n+s}x^s \right| < M_2 ;$$

evidently M_1 is independent of x and M_2 is likewise if $|x| \leqq K_1 < 1/2K$. Consequently the repeated series obtained from (13) by putting for x^m the expansion (10) is absolutely convergent if $|x|<|t|$ and $|x|<1/2K$, for then we have

$$\begin{aligned} \left| \sum_{n=m}^{\infty} b_{mn}U_n(x) \right| &< M_1 \sum_{n=m}^{\infty} |b_{mn}x^n| \\ &= M_1 |x|^m \sum_{s=0}^{\infty} |b_{m,m+s}x^s| \\ &< M_1 M_2 |x|^m. \end{aligned}$$

We may then interchange the orders of summation in this repeated series and obtain the expansion

$$1/(t-x) = \sum_{n=0}^{\infty} v_n(t)U_n(x),$$

where

$$v_n(t) = \sum_{s=0}^{n} b_{sn}/t^{s+1},$$

this expansion being valid for $|x|<|t|$ and $|x|<1/2K$.

If x is confined to the interior and boundary of a circle c with center at $x=0$ whose radius is less than $1/2K$ and t is confined to the boundary and exterior of a circle of radius greater than that of c, then the convergence of the expansion is obviously uniform with respect to both t and x.

We may now state the following theorem:

THEOREM I. *There exists a constant σ, $0<\sigma\leqq|q|^kR$, such that, if x and t are two complex variables satisfying the conditions*

$$|x| \leqq R_1 < \sigma,$$

and

$$|t| \geqq R_2 > R_1,$$

respectively, the expansion

$$1/(t-x) = \sum_{n=0}^{\infty} v_n(t)U_n(x)$$

is valid and converges uniformly with regard to both t and x.

5. **Expansion of $f(x)$.** Let $f(x)$ be single-valued and analytic on and within the circle C_2 of radius $R_2\leqq\sigma$ with center at $x=0$. Then by Cauchy's formula, for any x such that $|x|<R_2$ we have

$$f(x) = \frac{1}{2\pi i}\int_{C_2}\left\{\sum_{n=0}^{\infty} v_n(t)U_n(x)\right\}f(t)dt$$

$$= \sum_{n=0}^{\infty} a_nU_n(x)$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2} v_n(t)f(t)dt.$$

This expansion converges uniformly when $|x|\leqq R_1<R_2$. We may now state

THEOREM II. *Let $f(x)$ be single-valued and analytic on and within the circle C_2 of radius $R_2\leqq\sigma$ with center at $x=0$. Then the expansion*

$$f(x) = \sum_{n=0}^{\infty} a_nU_n(x), \tag{14}$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2} v_n(t)f(t)dt,$$

is valid if $|x|<R_2$ *and converges uniformly with respect to* x *when* $|x|\leqq R_1<R_2$.

6. **Relation to Maclaurin's expansion.** Let $f(x)$ have the Maclaurin expansion

$$f(x) = \sum_{m=0}^{\infty} b_m x^m.$$

Then

$$a_n = \frac{1}{2\pi i}\int_{C_2} v_n(t)\left\{\sum_{m=0}^{\infty} b_m t^m\right\}dt$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty} b_m \int_{C_2} t^m\left\{\sum_{s=0}^{n} b_{sn}/t^{s+1}\right\}dt$$

whence

$$a_n = \sum_{m=0}^{n} b_m b_{mn}.$$

7. **Analogue of descending power series expansion.** The expansion

$$1/(x-t) = \sum_{n=0}^{\infty} U_n(t)v_n(x)$$

is valid for $|t|<|x|$ and $|t|<\sigma$. Let Γ_1 be the circle $|x|=r_1\leqq\sigma$ and let Γ_2 be the circle $|t|=r_2<r_1$. Then if $g(x)$ is single-valued and analytic on and without Γ_2 and if $g(\infty)=0$, we have

$$g(x) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{g(t)dt}{(x-t)}$$

$$= \sum_{n=0}^{\infty} a_n' v_n(x),$$

where

$$a_n' = \frac{1}{2\pi i}\int_{\Gamma_1} U_n(t)g(t)dt,$$

this expansion being valid for $|x|>r_2$ and converging uniformly for $|x|\geqq r_1$. Hence we have proved

THEOREM III. *Let* $f(x)$ *be single-valued and analytic on and without the circle* Γ_2 *with center at* $x=0$ *and radius* $r_2<\sigma$. *Then the expansion*

$$f(x) = f(\infty) + \sum_{n=0}^{\infty} a_n' v_n(x), \tag{15}$$

where

$$a_n' = \frac{1}{2\pi i}\int_{\Gamma_1} U_n(t)\left\{f(t) - f(\infty)\right\}dt,$$

is valid for $|x|>r_2$ *and converges uniformly for* $|x|\geqq r_1>r_2$.

8. **Analogue of Laurent's series.** Let Γ_2 and C_2 be the circles with centers at $x=0$ and radii r_2 and R_2 respectively, with $r_2<R_2\leqq\sigma$, and let $f(x)$ be single-valued and analytic on the boundary and interior of the ring-shaped region S defined by the inequalities

$$0<r_2\leqq|x|\leqq R_2\leqq\sigma.$$

Then if x is any interior point of S, Laurent's theorem states that

$$f(x)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(t)dt}{(x-t)}+\frac{1}{2\pi i}\int_{C_2}\frac{f(t)dt}{(t-x)}.$$

Now in the first of these integrals

$$|t|<|x|<\sigma,$$

so that here we may put

$$1/(x-t)=\sum_{n=0}^{\infty}U_n(t)v_n(x),$$

and in the second

$$|x|<|t|\leqq\sigma,$$

so that in this integral we have

$$1/(t-x)=\sum_{n=0}^{\infty}v_n(t)U_n(x).$$

Therefore $f(x)$ has the expansion

$$f(x)=\frac{1}{2\pi i}\int_{\Gamma_1}\left\{\sum_{n=0}^{\infty}U_n(t)v_n(x)\right\}f(t)dt+\frac{1}{2\pi i}\int_{C_2}\left\{\sum_{n=0}^{\infty}v_n(t)U_n(x)\right\}f(t)dt.$$

From this last relation we may at once deduce the following theorem:

THEOREM IV. *Let $f(x)$ be single-valued and analytic on the boundary and interior of the ring-shaped region S defined by the inequalities*

$$0<r_2\leqq|x|\leqq R_2\leqq\sigma.$$

Then the expansion

$$f(x)=\sum_{n=0}^{\infty}a_nU_n(x)+\sum_{n=0}^{\infty}\alpha_nv_n(x), \tag{16}$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_2}v_n(t)f(t)dt,$$

and

$$\alpha_n = \frac{1}{2\pi i}\int_{\Gamma_1} U_n(t)f(t)dt\,,$$

is valid for every interior point of S and converges uniformly when x is in the region defined by the inequalities

$$r_2 < r_1 \leqq |x| \leqq R_1 < R_2.$$

If the Laurent expansion of $f(x)$ in the region S is

$$f(x) = \sum_{n=0}^{\infty} b_n x^n + \sum_{n=1}^{\infty} \beta_n x^{-n},$$

one readily finds that

$$a_n = \sum_{m=0}^{n} b_m b_{mn},$$

and

$$\alpha_n = \beta_{n+1} + \sum_{m=1}^{\infty} c_{nm}\beta_{n+m+1}.$$

9. **Orthogonality properties.** We shall now obtain the following formulas:

$$\begin{aligned}
&\int_D U_m(x)U_n(x)dx = 0 && (m = n \text{ and } m \neq n),\\
&\int_D v_m(x)v_n(x)dx = 0 && (m = n \text{ and } m \neq n),\\
&\int_D U_m(x)v_n(x)dx = 0 && (m \neq n),\\
&\int_D U_m(x)v_m(x)dx = 2\pi i,
\end{aligned} \tag{17}$$

where the integration is taken in a positive direction along D, a closed curve which encircles the point $x=0$ once and lies within the circle $|x|=\sigma$. The first of these formulas follows immediately from the fact that $U_m(x)$ and $U_n(x)$ are analytic on and within D. The second one holds since the product $v_m(x)v_n(x)$ consists of terms in x^{-2} and lower powers of x. The third relation is evident if $m>n$, since then there is no term in x^{-1} in the product $U_m(x)v_n(x)$. If $m<n$ the coefficient of x^{-1} in $U_m(x)v_n(x)$ is

$$b_{m,m+s} + b_{m+1,m+s}\,c_{m1} + b_{m+2,m+s}\,c_{m2} + \cdots + b_{m+s,m+s}\,c_{ms},$$

where s has been put for $n-m$. To see that this coefficient vanishes, write

the second of equations (11) for $s=1, 2, 3, \cdots, s$ and solve this system for $b_{m,m+s}$. This gives the result

$$b_{m,m+s} = -\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & c_{m1} \\ c_{m+1,1} & 1 & 0 & \cdots & 0 & c_{m2} \\ c_{m+1,2} & c_{m+2,1} & 1 & \cdots & 0 & c_{m3} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ c_{m+1,s-1} & c_{m+2,s-2} & c_{m+3,s-3} & \cdots & c_{m+s-1,1} & c_{ms} \end{vmatrix}.$$

By expanding this determinant according to the elements of the last column, we readily obtain the relation

$$b_{m,m+s} = -(b_{m+1,m+s}c_{m1} + b_{m+2,m+s}c_{m2} + \cdots + b_{m+s,m+s}c_{ms}).$$

To verify the last of formulas (17) is it only necessary to note that the coefficient of x^{-1} in $U_m(x)v_m(x)$ is unity.

By means of these formulas it is possible to obtain directly the coefficients in the expansions (14), (15), and (16). If, for example, we have an expansion of the type

$$f(x) = \sum_{n=0}^{\infty} a_n U_n(x) + \sum_{n=0}^{\infty} \alpha_n v_n(x) \tag{16}$$

and let C be any curve in the ring-shaped region in which this expansion is valid and including the point $x=0$ once, then by multiplying both members of (16) by $U_n(x)$ and integrating term by term along C with respect to x, we have

$$\alpha_n = \frac{1}{2\pi i}\int_C U_n(x)f(x)dx.$$

Similarly, multiplying both members by $v_n(x)$ and integrating, we get

$$a_n = \frac{1}{2\pi i}\int_C v_n(x)f(x)dx.$$

10. **The case of linear coefficients.** In the special case in which the coefficients $A_0(x), A_1(x), \cdots, A_k(x)$ are linear, equation (2) becomes

$$A_i(x) = a_{i0} + a_{i1}x \qquad (i = 0,1,2, \cdots, k),$$

while equation (4) is unchanged. The coefficients c_{ns} and $b_{n,n+s}$, which may be readily calculated, are given by the equations

$$c_{ns} = \prod_{i=0}^{s-1} \rho_{n+i}/(\lambda_{n+i+1} - \lambda_n), \qquad s > 0,$$

$$b_{nn} = 1,$$

$$b_{n,n+s} = \prod_{i=0}^{s-1} \rho_{n+i}/(\lambda_{n+i} - \lambda_{n+s}), \qquad s > 0,$$

where

$$\rho_n = \sum_{j=0}^{k} a_{j1} q^{(k-j)(n+\nu)}.$$

The radius R of the circle within which the series

$$x^{n+\nu}\left(1 + \sum_{s=1}^{\infty} c_{ns} x^s\right)$$

represents a solution of the q-difference equation is $|a_{00}/a_{01}|$ and it is not hard to show that the expansion (10), (11) is valid for $|x| < |q|^k |a_{00}/a_{01}|$, so that here the σ of Theorems I, II, III, IV is equal to $|q|^k |a_{00}/a_{01}|$; that is, it is equal to the radius of the circle of convergence of the power series expansion of $U_n(x)$.

In the case in which the functions $A_0(x), A_1(x), \cdots, A_k(x)$ are linear functions of x, the functions $v_n(t)$ satisfy a simple homogeneous kth order q-difference equation with linear coefficients if the relation

$$T \equiv a_{01} q^{k(\nu-1)} + a_{11} q^{(k-1)(\nu-1)} + a_{21} q^{(k-2)(\nu-1)} + \cdots + a_{k1} = 0$$

is satisfied. For from the equation

$$x^\nu/(t - x) = \sum_{n=0}^{\infty} v_n(t) u_n(x)$$

and equation (1) it follows that

$$q^{k\nu} x^\nu (a_{00} + a_{01}x)/(t - q^k x) + q^{(k-1)\nu} x^\nu (a_{10} + a_{11}x)/(t - q^{k-1}x)$$
$$+ \cdots + x^\nu (a_{k0} + a_{k1}x)/(t - x) = -\sum_{n=0}^{\infty} \lambda_n v_n(t) u_n(x),$$

or

$$-Tx^\nu + x^\nu (a_{00} q^{k(\nu-1)} + a_{01} t q^{k(\nu-2)})/(tq^{-k} - x)$$
$$+ x^\nu (a_{10} q^{(k-1)(\nu-1)} + a_{11} t q^{(k-1)(\nu-2)})/(tq^{-k+1} - x)$$
$$+ \cdots + x^\nu (a_{k0} + a_{k1}t)/(t - x) = -\sum_{n=0}^{\infty} \lambda_n v_n(t) u_n(x),$$

whence

$$\sum_{n=0}^{\infty} u_n(x)\{(a_{00}q^{k(\nu-1)} + a_{01}tq^{k(\nu-2)})v_n(t/q^k) + (a_{10}q^{(k-1)(\nu-1)} + a_{11}tq^{(k-1)(\nu-2)}v_n(t/q^{k-1}) + \cdots + (a_{k0} + a_{k1}t)v_n(t)\} = -\sum_{n=0}^{\infty} \lambda_n v_n(t)u_n(x).$$

Consequently the functions $v(t)$ satisfy the equation

$$A_0'(t)v_n(t/q^k) + A_1'(t)v_n(t/q^{k-1}) + \cdots + A_{k-1}'(t)v_n(t/q) + (\lambda_n + A_k'(t))v_n(t) = 0,$$

where

$$A_i'(t) = a_{i0}q^{(k-i)(\nu-1)} + a_{i1}q^{(k-i)(\nu-2)}t \quad (i = 0,1,2,\cdots,k).$$

If the relation $T=0$ is not satisfied, $v_n(t)$ evidently satisfies a non-homogeneous equation.

More generally, if the functions $A_0(x)$, $A_1(x)$, $\cdots$, $A_k(x)$ are polynomials, each of degree d or less, $v_n(t)$ satisfies a homogeneous linear q-difference equation of order k with coefficients $A_i'(t)$, where $A_i'(t)$ is a polynomial of the same degree as $A_i(x)$, provided that the $d(d+1)/2$ equations

$$\sum_{i=0}^{k} a_{ij}q^{(k-i)(\nu-\mu)} = 0 \qquad (j = 1,2,3,\cdots,d\,;\ \mu = 1,2,3,\cdots,j)$$

are satisfied. If some one of these equations fails to hold, $v_n(t)$ satisfies a non-homogeneous equation.

11. **The problem when** $|q|<1$. If, instead of supposing that $|q|>1$, we assume that $|q|<1$, the problem is essentially the same as before with the rôles of the points infinity and zero interchanged. If in equation (1) we replace x by $1/y$ and q by $1/p$ and put $A_i(1/y)=B_i(y)$ and $u_n(1/y)=w_n(y)$, we have

$$B_0(y)w_n(p^k y) + B_1(y)w_n(p^{k-1}y) + \cdots + B_{k-1}(y)w_n(py) + (\lambda_n + B_k(y))w_n(y) = 0,$$

where $|p|<1$, the coefficients $B_i(y)$ are analytic at $y=\infty$, and $B_0(\infty)\neq 0$. Consequently, if we suppose that in equation (1) the coefficients $A_i(x)$ are analytic at $x=\infty$, $A_0(\infty)$ being different from zero, and that $|q|<1$, we may obtain expansions of arbitrary functions analytic at infinity in terms of solutions

$$u_{-n}(x) = x^{-n-\nu}\left(1 + \sum_{s=1}^{\infty} c_{-n,s}x^{-s}\right),$$

corresponding to the expansions of Theorems II, III, IV, and V. These expansions may be obtained directly by expanding $x^{-m-\nu}$ in terms of the $u_{-n}(x)$, substituting these expansions in the identity

$$x^{-\nu}/(x-t) = \sum_{m=0}^{\infty} t^m/x^{m+\nu+1},$$

and applying Cauchy's theorem as before.

If the coefficients $A_i(x)$ are analytic at both zero and infinity, as, for example, when each function $A_i(x)$ is the quotient of one polynomial, $E_i(x)$, divided by another, $F_i(x)$, of the same degree, $F_i(0)$ being different from zero, there are expansions in both cases, $|q|<1$ and $|q|>1$.

III. Expansions of analytic functions of several variables

12. The expansion theorems obtained in Section II may easily be extended to apply to functions of several variables. If x_1, t_1, and x_2, t_2 are two pairs of complex variables in the x_1- and x_2-planes respectively, such that

$$\begin{aligned} &|x_1| \leqq R_1, \qquad |t_1| \geqq R_1', \qquad R_1 < R_1' \leqq \sigma, \\ &|x_2| \leqq R_2, \qquad |t_2| \geqq R_2', \qquad R_2 < R_2' \leqq \sigma, \end{aligned} \tag{18}$$

then by Theorem I the expansions

$$1/(t_1 - x_1) = \sum_{n_1=0}^{\infty} v_{n_1}(t_1) U_{n_1}(x_1),$$

and

$$1/(t_2 - x_2) = \sum_{n_2=0}^{\infty} v_{n_2}(t_2) U_{n_2}(x_2)$$

are valid and each converges uniformly with respect to both variables. Multiplying the two series together, we obtain their product

$$\frac{1}{(t_1 - x_1)} \frac{1}{(t_2 - x_2)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} v_{n_1}(t_1) v_{n_2}(t_2) U_{n_1}(x_1) U_{n_2}(x_2),$$

which is absolutely and uniformly convergent.

Let C_1 and C_2 be the contours formed by the circles $|x_1| = R_1'$ and $|x_2| = R_2'$ and let $f(x_1, x_2)$ be a single-valued analytic function of the two variables x_1 and x_2 when x_1 is on and within C_1 and x_2 is on and within C_2. Then we have that

$$f(x_1, x_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \frac{f(t_1,t_2)}{(t_1 - x_1)(t_2 - x_2)}\, dt_2 dt_1$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} v_{n_1}(t_1) v_{n_2}(t_2)\, U_{n_1}(x_1)\, U_{n_2}(x_2) \right\} f(t_1,t_2)\, dt_2 dt_1,$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1 n_2} U_{n_1}(x_1)\, U_{n_2}(x_2),$$

where

$$a_{n_1 n_2} = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} v_{n_1}(t_1) v_{n_2}(t_2) f(t_1,t_2)\, dt_2 dt_1.$$

This expansion is valid and converges uniformly with respect to x_1 and x_2 when $|x_1| \leqq R_1$ and $|x_2| \leqq R_2$.

In precisely the same manner we may prove the following theorem:

THEOREM V. *Let $f(x_1, x_2, \cdots, x_\lambda)$ be a function of $x_1, x_2, \cdots, x_\lambda$ which is single-valued and analytic when $x_1, x_2, \cdots, x_\lambda$ are on and within the respective contours $C_1, C_2, \cdots, C_\lambda$ formed by the circles $|x_1| = R_1' \leqq \sigma$, $|x_2| = R_2' \leqq \sigma, \cdots, |x_\lambda| = R_\lambda' \leqq \sigma$. Then the expansion*

$$f(x_1, x_2, \cdots, x_\lambda) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_\lambda=0}^{\infty} a_{n_1 n_2 \ldots n_\lambda} U_{n_1}(x_1)\, U_{n_2}(x_2) \cdots U_{n_\lambda}(x_\lambda),$$

where

$$a_{n_1 n_2 \ldots n_\lambda} = \left(\frac{1}{2\pi i}\right)^\lambda \int_{C_1}\int_{C_2} \cdots \int_{C_\lambda} v_{n_1}(t_1) v_{n_2}(t_2) \cdots v_{n_\lambda}(t_\lambda) f(t_1, t_2, \cdots, t_\lambda)\, dt_\lambda \cdots dt_1,$$

is valid for $|x_1| < R_1'$, $|x_2| < R_2'$, $\cdots$, $|x_\lambda| < R_\lambda'$ and converges uniformly for

$$|x_1| \leqq R_1 < R_1', \; |x_2| \leqq R_2 < R_2', \cdots, |x_\lambda| \leqq R_\lambda < R_\lambda'.$$

Evidently Theorems III and IV may be generalized in like manner.

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SOME PROPERTIES OF LIMITED CONTINUA, IRREDUCIBLE BETWEEN TWO POINTS*

BY

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1. In reading papers on the theory of point aggregates one frequently meets with the continuum C whose points are defined as follows: when $-1 \leqq x < 0$ or $0 < x \leqq 1$, $y = \sin(1/x)$; when $x = 0$, $-1 \leqq y \leqq 1$. This is perhaps the most commonly cited example of a limited continuum irreducible between two points and having a continuum of condensation. If we regard the concept of connectedness im kleinen as the analogue for continua of continuity in functions, we notice that the properties of this continuum resemble those of pointwise discontinuous functions. The points where the continuum is connected im kleinen form a set of the secondary category with respect to C, while those of the second genre† form a set of the first category.

The question at once arises as to whether this similarity is of a general character or is merely due to the nature of the example cited, and suggests that a study of limited continua irreducible between two points with special reference to the oscillation at the various points would be of interest. This problem has been discussed in a paper by H. Hahn,‡ who has shown that such a continuum is the sum of a set of sub-continua known as "prime parts," no two of which have common points. However, the fact that in many cases a prime part itself can be subdivided indicates that the subject has not been exhausted and it is the purpose of this paper to present some further results along this line.

The first half of the paper (§§4–15) is devoted to the general properties of the oscillation of a limited irreducible continuum. In the second half the properties such continua have when the points of the first genre are everywhere dense are treated, and in particular it is shown that in this case there is a correspondence between the points of such a continuum and those of a linear segment analogous to that between the variables y and x when $y = f(x)$ is a pointwise discontinuous function of a certain type.

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† A point of a continuum is of the first or second genre according as the oscillation of the continuum at the point is zero or not. See S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, p. 170.

‡ H. Hahn, *Über irreduzible Kontinua*, Wiener Sitzungsberichte, vol. 130, pp. 217–250.

2. In a previous paper,* of which this is in one sense a continuation, two concepts were introduced which will be used throughout this work. The first was an auxiliary function $\tau(x)$, suggested as a definition of oscillation alternative to that of Mazurkiewicz.† This is defined as follows. Let c be a point of the continuum A, let $V(c)$ denote those points of A whose distance from c is less than δ and let C_δ be any sub-continuum of A containing $V(c)$. Then the lower bound of the diameters of all such sub-continua C_δ for all values of $\delta>0$ is denoted by $\tau(c)$. The function $\tau(c)$ will be called the oscillation of A at c.

The second concept was that of *oscillatory sets*. Let $\delta_1>\delta_2>\cdots$, $\delta_i\to 0$, and let $\{C_{\delta_i}\}$ be a monotone decreasing sequence of sub-continua of A irreducible about $V_{\delta_i}(c)$. Then $C(c)=Dv[C_{\delta_i}]$ is called an oscillatory set of A about c. It is obvious that, if A is limited, each oscillatory set is a continuum.

Among the general properties of these concepts derived in the paper referred to, the following will be used.

(*a*) For limited continua there exist one or more oscillatory sets at each point and $\tau(c)$ is the lower bound of the diameters of the oscillatory sets of A about c.

(*b*) If $\tau(c)=0$, the point c is one of the oscillatory sets of A about c.

(*c*) The function $\tau(c)$ is related to Mazurkiewicz' oscillatory function $\sigma(c)$ by the inequalities $\sigma(c)\leqq\tau(c)\leqq 2\sigma(c)$.

3. **Notation.** The ordinary notation of the theory of aggregates is employed, with the following modifications.

The notation $A\subset B$ means that A is a *real* part of B. If A is a part of B and may be identical with B, we write $A\subseteq B$.

If A is the common part of the system of aggregates $\{C\}$, we write $A=Dv[C]$.

If A contains every element of each of the system of aggregates $\{C\}$ and no other elements, we write $A=U[C]$.

When the notation $A=B+C$ is used, it will be understood unless expressly stated to the contrary that $B\cdot C=0$.

The phrase "irreducible continuum ab" means a continuum which is irreducible between the points a and b.

The symbol $S_\delta(c)$ denotes an open sphere (or hypersphere) of center c and radius δ. If c is a point of the aggregate A, the symbol $V_\delta(c)$ denotes the the subset of points of A each of which has a distance from c less than δ.

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* W. A. Wilson, *On the oscillation of a continuum*, these Transactions, vol. 27, pp. 429–440.

† Loc. cit.

Then $V_\delta(c) = A \cdot S_\delta(c)$ and $V(c)$ is a region relative to A. In like manner, if E is a closed subset of A, the symbol $V_\delta(E)$ denotes the subset of points of A each of which has a distance from E less than δ.

4. In the following chain of six theorems, of which the first three are essentially lemmas, certain general properties of a limited irreducible continuum ab are developed. It is shown that ab has about each point a unique oscillatory set of diameter $\tau(c)$, that at no point x of an oscillatory set of diameter greater than zero is $\tau(x) = 0$, and that $\tau(c)$ is identical with Mazurkiewicz' oscillatory function $\sigma(c)$.

THEOREM. *Let ab be a limited irreducible continuum. Let A and B be true sub-continua of ab containing a and b respectively and let $A \cdot B = 0$. Then there is one and only one sub-continuum K of ab which is irreducible between A and B, and $K = \overline{ab - (A + B)}$.*

Proof. By a continuum K irreducible between A and B we mean a continuum containing points of both A and B such that no true sub-continuum contains points of both these sets. Let $L = ab - (A + B)$ and consider $\bar{L}$. Since ab is a continuum,

$$ab = A + \bar{L} + B, \qquad A \cdot \bar{L} \neq 0, \qquad \bar{L} \cdot B \neq 0.$$

Now there is a partition of L into two closed sets C and D such that $C \cdot D = C \cdot B = D \cdot A = 0$, $A \cdot C \neq 0$, and $B \cdot D \neq 0$; or there is not. In the former case, $ab = (A + C) + (B + D)$ and $(A + C)(B + D) = 0$, which is contrary to the fact that ab is a continuum.

In the latter case an easy generalization of two theorems of Janiszewski* shows that $\bar{L}$ contains a continuum K joining A and B. Since ab is irreducible, $K \supseteq ab - (A + B) = L$, whence $K \supseteq \bar{L}$. Thus $\bar{L}$ is a continuum joining A and B. Finally, any sub-continuum of ab irreducible between A and B must contain L and consequently $\bar{L}$, since ab is irreducible. Hence the theorem.†

COROLLARY. *Let ab be a limited irreducible continuum and A be a true sub-continuum containing a but not b. Then there is one and only one sub-continuum K of ab which is irreducible between A and b, and $K = \overline{ab - A}$.*

5. THEOREM. *Let C be a limited continuum. Let a and c be points of C and let $\delta < \text{Dist}\ (a, c)$. Let A be the saturated sub-continuum of C containing a, but no point of $V_\delta(c)$. Then at least one point of A is a limiting point of $V_\delta(c)$.*

* Z. Janiszewski, *Sur les continus irreductibles entre deux points*, Journal de l'Ecole Polytechnique, ser. 2, vol. 16 (1912), pp. 109–111.

† The proof here given was suggested by the referee in lieu of a longer one in the paper as originally submitted.

Proof. Let S_δ denote the interior of a sphere of center c and radius δ. Let Z denote the whole of the n-way euclidean space containing C, and G_δ be the complement of S_δ. Then G_δ is closed and

$$Z = G_\delta + S_\delta.$$

Let $\sigma < \delta$ and let V denote those points of Z whose distance from the closed set A is not greater than σ. Then V is closed and does not contain c. Also

$$V = V \cdot G_\delta + V \cdot S_\delta.$$

By a well known theorem of Janiszewski* there is a sub-continuum K of C joining a point z of A to a point y of Front V and lying wholly in V. If $K \subseteq V \cdot G_\delta$ then K would contain no point of S_δ, while $K \cdot A \neq 0$. This would make $K \subseteq A$ contrary to the fact that y is not a point of A. Thus K contains at least one point in $V \cdot S_\delta$. But $K \subset V$. Hence at least one point of C in S_δ has a distance from A not greater than σ.

As this is true for every σ, we have a sequence of points of $V_\delta(c) = C \cdot S_\delta$ whose distances from A converge to zero. These therefore have at least one limiting point on A.

6. THEOREM. *Let ab be a limited irreducible continuum and c be a point of ab. Then there is a unique sub-continuum of ab irreducible about any $V_\delta(c)$.*

Proof. Let C_δ be any sub-continuum of ab irreducible about $V_\delta(c)$. There are several cases to consider.

I. If $V_\delta(c)$ contains both a and b, C_δ obviously equals ab.

II. If $V_\delta(c)$ contains b but not a, let A be the saturated sub-continuum of ab containing a but no point of $V_\delta(c)$ and let $\bar{L} = \overline{ab - A}$.

Since $C_\delta \supseteq V_\delta(c)$, $C_\delta \supseteq \bar{V}_\delta(c)$. Then, by §5, $A \cdot C_\delta \neq 0$. As C_δ contains b and a point of A, we have by §4, Corollary,

$$C_\delta \supseteq \bar{L}. \tag{1}$$

But

$$\bar{L} \supset ab - A \supseteq V_\delta(c). \tag{2}$$

Since C_δ is irreducible about $V_\delta(c)$, relations (1) and (2) give

$$C_\delta = \bar{L}.$$

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* See reference under § 4, p. 100.

III. If $V_\delta(c)$ contains a but not b, we have essentially the same situation as in Case II.

IV. If $V_\delta(c)$ contains neither a nor b, let A be defined as in Case II and let B have the same relation to the point b, while $\overline{L}=\overline{ab-(A+B)}$. As in Case II, $A\cdot C_\delta\neq 0$ and $B\cdot C_\delta\neq 0$. Hence, by §4,

$$C_\delta \supseteq \overline{L}. \tag{3}$$

But

$$\overline{L} \supset ab-(A+B) \supseteq V_\delta(c). \tag{4}$$

Then relations (3) and (4) give

$$C_\delta = \overline{L}.$$

7. THEOREM. *Let ab be a limited irreducible continuum and let c be a point of ab. Then the oscillatory set of ab about c is unique and its diameter is $\tau(c)$, and, if $\tau(c)=0$, it is the point c itself.*

Proof. Let $0<\delta<\eta$, and let C_δ and C_η be the sub-continua of ab irreducible about $V_\delta(c)$ and $V_\eta(c)$ respectively. Obviously $V_\delta(c)\subseteq V_\eta(c)$ and hence $V_\delta(c)\subseteq C_\eta$.

Then C_η has a sub-continuum D irreducible about $V_\delta(c)$. Since $D\subset C_\eta$,

$$D\subseteq ab. \tag{1}$$

But C_δ is the only sub-continuum of ab irreducible about $V_\delta(c)$. Hence (1) gives $C_\delta=D$ and $C_\delta\subseteq C_\eta$.

This result shows that if $\{\delta_i\}$ and $\{\eta_i\}$ are monotone decreasing sequences converging to zero, $Dv[C_{\delta_i}]=Dv[C_{\eta_i}]$. Hence the oscillatory set is unique. That its diameter is $\tau(c)$ follows from the fact that for continua in general $\tau(c)$ is equal to the minimum diameter of all the oscillatory sets about c. (See §2.) Then if $\tau(c)=0$, the oscillatory set can contain no other point than c itself.

For continua in general the oscillatory sets are not unique, as is shown in an example given in the paper referred to in §2. The reader should note the analogy of the oscillatory set to the aggregate of limiting values of a one-valued function of a real variable $f(x)$ as $x\to a$. It is to the above theorem that the oscillatory sets of a limited irreducible continuum owe most of their value. A point of the second genre lies on a continuum of condensation and in simple cases there is a greatest such continuum, but this is not always the case. On the other hand the oscillatory set of each point is unique.

8. THEOREM. *Let ab be a limited irreducible continuum. Then for each point c of ab, $\tau(c)=\sigma(c)$.*

Proof. It has been shown elsewhere (see §2) that

$$\sigma(c) \leqq \tau(c). \tag{1}$$

Let $c \neq a$, $c \neq b$. Then by definition* of $\sigma(c)$ *any* two points x and y of ab lying in a closed sphere $\bar{S}_\delta(c)$ whose center is c and whose radius δ is sufficiently small can be joined by a sub-continuum $C(x, y)$ of ab such that

$$\text{Diam } C(x,y) < \sigma(c) + \epsilon, \tag{2}$$

where ϵ is any positive quantity.

Now let $S_\delta(c)$ denote the interior of $\bar{S}_\delta(c)$, and let δ be so small that $\bar{S}_\delta(c)$ contains neither a nor b. Let A and B be the saturated sub-continua of ab containing a and b respectively, but no point of $V_\delta(c) = ab \cdot S_\delta(c)$. By §5, A contains a point x, and B a point y, on Front $\bar{S}_\delta(c)$, and these points satisfy relation (2). Then

$$ab = A + C(x,y) + B, \qquad A \cdot C(x,y) \neq 0, \qquad B \cdot C(x,y) \neq 0.$$

Since $A \cdot V_\delta(c) = B \cdot V_\delta(c) = 0$, we have

$$V_\delta(c) \subseteq C(x,y).$$

Then relation (2) gives

$$\tau(c) \leqq \sigma(c) + \epsilon.$$

This with relation (1) gives the theorem for the case under consideration. Similar reasoning applies to the cases that $c=a$ or $c=b$.

9. THEOREM. *Let ab be a limited irreducible continuum. Let c be a point of the second genre of ab and let $C(c)$ be the oscillatory set of ab about c. Then no point of $C(c)$ is of the first genre.*

Proof. Assume that there is a point x of $C(c)$ at which $\tau(x)=0$, and let $\eta = \text{Dist } (x, c)$. Then for any positive $\epsilon < \eta$ there is a $\delta > 0$, such that there is a sub-continuum D of ab having the property

$$V_\delta(x) \subseteq D \subseteq V_\epsilon(x). \tag{1}$$

Let c be different from a and b and let δ be so small that $V_\delta(x)$ contains neither a nor b. Let A and B be the saturated sub-continua of ab containing a and b respectively, but no point of $V_\delta(x)$. Then by §5 and relation (1)

$$ab = A + D + B, \qquad A \cdot D \neq 0, \qquad B \cdot D \neq 0.$$

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* See S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, p. 170.

Since $\epsilon<\eta$, c is not a point of D; suppose that c is a point of A. Then A has a sub-continuum irreducible about any $V_\sigma(c)$, if σ is so small that $V_\sigma(c)$ contains no point of D or of B. Since $A \subset ab$, this is the sub-continuum of ab irreducible about $V_\sigma(c)$ by § 6. Hence we have $C(c) \subseteq A$. This is a contradiction, since x is not a point of A.

The cases where c coincides with a or b are treated in a like manner.

Remark. The above theorem shows that the oscillatory set about a point c is a part of the prime part containing c. That it need not be identical with the prime part is seen from the example suggested in § 23.

10. It would be natural to surmise from § 9 that, if $C(c)$ is the oscillatory set of ab about c, then $C(c)$ is the oscillatory set about each point z on $C(c)$ and that $\tau(z)=\tau(c)$. This, however, is not true, as the following example shows. Let ab be the continuum whose points are defined as follows: for $x=0$, $-1\leqq y\leqq 1$; for $0<x\leqq 1$, $y=\sin^2(1/x)$; for $-1\leqq x<0$, $y=-\sin^2(1/x)$. The oscillatory set of ab about the point $(0, 0)$ is the segment of the y-axis between $(0, -1)$ and $(0, 1)$ and the oscillation is 2. For the point $(0, \frac{1}{2})$ the oscillatory set is the segment joining $(0, 0)$ and $(0, 1)$ and the oscillation is 1.

In order to investigate this question more fully it is necessary to consider the point set remaining when the oscillatory set about the point c is subtracted from the continuum ab. The inference that the remainder is two semi-continua, one containing a and the other b (or one semi-continuum if $C(c)$ contains either a or b) is seen to be false for the point $(0, \frac{1}{2})$ in the example just given. The difficulty is that $(0, \frac{1}{2})$ is a limiting point of only one of the semi-continua containing the end points of ab. The inference is borne out for $(0, 0)$, however, as this point is a limiting point of both semi-continua.

The following notation will now be used. If x is a point of ab, the oscillatory set of ab about x will be denoted by $X(x)$, or simply X. The saturated semi-continua of $ab-X$ containing a and b respectively will be denoted by X_a and X_b. If b is a point of X, obviously $X_b=0$; if a is a point of X, $X_a=0$. It is evident that

$$X_a + X + X_b \subseteq ab \quad \text{and} \quad X_a \cdot X_b = 0. \tag{1}$$

By a theorem of Janiszewski* we know that $X\cdot\overline{X}_a\neq 0$ and $X\cdot\overline{X}_b\neq 0$; hence

$$ab = \bar{X}_a + X + \bar{X}_b, \qquad X\cdot\bar{X}_a \neq 0, \qquad X\cdot\bar{X}_b \neq 0. \tag{2}$$

Thus $\overline{X}_a+X$ and $X+\overline{X}_b$ are continua. Analogous theorems hold, of course, for the cases that $X_a=0$ and $X_b=0$.

* See reference under § 4, p. 123.

It is also easy to show that

$$\bar{X}_a \cdot X_b = 0 \quad \text{and} \quad X_a \cdot \bar{X}_b = 0. \tag{3}$$

For, let A and B be the saturated sub-continua of ab containing a and b but no points of $V_\delta(x)$. Then $A \cdot X_b = 0$ and $B \cdot X_a = 0$, since ab is irreducible. But

$$ab = A + X_\delta(x) + B, \qquad A \cdot X_\delta(x) \neq 0, \qquad B \cdot X_\delta(x) \neq 0.$$

Then

$$X_a \subseteq A + X_\delta(x),$$

and

$$\bar{X}_a \cdot X_b \subseteq A \cdot X_b + X_b \cdot X_\delta(x) = X_b \cdot X_\delta(x).$$

Hence

$$\bar{X}_a \cdot X_b \subseteq Dv[X_b \cdot X_\delta(x)] = X \cdot X_b = 0.$$

11. THEOREM. *Let C be the oscillatory set of the limited irreducible continuum ab about one of its points c and let C contain neither a nor b. Let c be a common limiting point of C_a and C_b. Then $ab = C_a + C + C_b$.*

Proof. Owing to the hypothesis regarding c any sub-continuum $C_\delta(c)$ of ab irreducible about $V_\delta(c)$ contains points of C_a and C_b. Hence

$$ab = C_a + C_\delta(c) + C_b, \qquad C_a \cdot C_\delta(c) \neq 0, \qquad C_b \cdot C_\delta(c) \neq 0,$$

or

$$C_\delta(c) \supseteq ab - (C_a + C_b).$$

Therefore

$$C \supseteq ab - (C_a + C_b). \tag{1}$$

But

$$C_a + C + C_b \subseteq ab, \qquad C \cdot C_a = 0, \qquad C \cdot C_b = 0,$$

whence

$$C \subseteq ab - (C_a + C_b). \tag{2}$$

Relations (1) and (2) give the theorem.

COROLLARY 1. *Let C be the oscillatory set of the limited irreducible continuum ab about the point c. Let a be a point of C and c be a limiting point of C_b. Then $ab = C + C_b$.*

COROLLARY 2. *Under the conditions of the above theorem the sets $C_a + C$ and $C + C_b$ are continua.*

For $\bar{C}_a C_b = 0$ by § 10, relation (3).

COROLLARY 3. *If* $\tau(c)=0$, $ab=C_a+c+C_b=\overline{C}_a+C_b=C_a+\overline{C}_b$.

For by § 10, relation (2), c is a common limiting point of C_a and C_b.

COROLLARY 4. *Under the conditions of the above theorem, C is a continuum of condensation of ab.*

For c is a point of $\overline{C}_a\cdot\overline{C}_b$. Hence $ab=\overline{C}_a+\overline{C}_b$. Therefore $C\subset\overline{C_a+C_b}$.

Analogous theorems hold for the oscillatory sets of the points a and b.

12. From § 11, Corollary 3, we can easily deduce the following well known properties of irreducible continua.

(*a*) If c is a point of the first genre of the limited irreducible continuum ab, then c divides ab into two unique irreducible continua ac and cb, and $ac\cdot cb=c$.

(*b*) If x and y are two points of the first genre of the limited irreducible continuum ab, then x and y divide ab into three unique irreducible continua; if x is a point of ay, $ab=ax+xy+yb$, and $ax\cdot xy=x$, $xy\cdot yb=y$, and $ax\cdot yb=0$.

(*c*) If c is a point of the first genre of the limited irreducible continuum ab and x is a point of ac, then every irreducible sub-continuum ax of ab is a real part of ac and every xb contains cb as a real part.

It is also evident that analogous theorems can be deduced from the partition $ab=C_a+C+C_b$ of § 11. These we shall not go into, but we need the two theorems immediately following.

13. THEOREM. *Let X be the oscillatory set of the limited irreducible continuum ab about the point x and let X contain neither a nor b. Let c be a common limiting point of* X_a *and* X_b. *Then the oscillatory set C about c contains X.*

Proof. Any sub-continuum $C_\delta(c)$ of ab irreducible about $V_\delta(c)$ contains points of both X_a and X_b. Hence

$$C_\delta(c)\supseteq ab-(X_a+X_b). \tag{1}$$

But

$$X_a+X+X_b\subset ab,$$

or

$$X\subseteq ab-(X_a+X_b). \tag{2}$$

Relations (1) and (2) give

$$C_\delta(c)\supseteq X.$$

Hence

$$C\supseteq X.$$

Analogous theorems hold for the oscillatory sets of a and b.

14. THEOREM. *Let ab be a limited irreducible continuum. Let c be a point of ab and let C be the oscillatory set of ab about c. Let c be a common limiting point of the saturated semi-continua C_a and C_b of $ab-C$. Let x be any point of C and X be the oscillatory set of ab about x. Then $X \subseteq C$.*

Proof. Let $\{\delta_i\}$ and $\{\delta_k\}$ be two decreasing sequences of positive numbers converging to zero. Let C_i be the sub-continuum of ab irreducible about $V_{\delta_i}(c)$, while D_k has the corresponding meaning for $V_{\delta_k}(x)$. Assume that C contains neither a nor b.

Since c is a limiting point of both C_a and C_b, for every i there is a sub-continuum A_i of C_a joining a to a point of $V_{\delta_i}(c)$, and likewise a B_i of analogous properties. Hence

$$ab = A_i + C_i + B_i, \qquad A_i \cdot C_i \neq 0, \qquad C_i \cdot B_i \neq 0.$$

Since $C \cdot A_i = C \cdot B_i = 0$, $V_{\delta_k}(x)$ contains no point of either A_i or B_i if k is sufficiently great. Thus $V_{\delta_k}(x) \subseteq C_i$ and so for every i there is a k such that

$$D_k \subseteq C_i.$$

Since $X = Dv[D_k]$ and $C = Dv[C_i]$, this gives at once

$$X \subseteq C.$$

The special cases where C contains a or b are treated in like manner.

COROLLARY. *If in the above theorem x is a common limiting point of C_a and C_b, then the oscillatory set X of ab about x is identical with C.*

15. The example of § 10 shows that if a point x lies on an oscillatory set $C(c)$, the oscillatory set $X(x)$ may form only a part of $C(c)$. In the example mentioned the point $(0, \frac{1}{2})$ is an instance of this. This fact, coupled with § 14, suggests setting apart as a separate class the oscillatory sets satisfying the hypotheses of that theorem. Accordingly we have the definition

If $C(c)$ is the oscillatory set of the limited irreducible continuum ab about a point c, we shall call $C(c)$ complete when it contains neither a nor b and c is a common limiting point of the saturated semi-continua C_a and C_b of $ab-C(c)$. Likewise, *if c is identical with a or b and c is a limiting point of C_b or C_a, respectively, we shall call $C(c)$ complete.*

In addition to the properties already derived in §§ 11-14, we have the following general theorem.

Let ab be a limited irreducible continuum and the oscillatory set $C(c)$ be complete. Let x be a point of C_a or C_b. Then the oscillatory set $X(x)$ is a part of C_a or C_b, respectively.

Proof. To fix the ideas let x be a point of C_a. Then it follows from the definition of oscillatory sets that there is a $\delta>0$ so small that, if C_δ is the sub-continuum of ab irreducible about $V_\delta(c)$, C_δ does not contain x. Since $C(c)$ is complete, there is a sub-continuum E of C_a containing a and a point in $V_\delta(c)$. Then

$$ab = E + C_\delta + C_b \text{ and } E \cdot C_b = 0, \qquad E \cdot C_\delta \neq 0, \qquad C_\delta \cdot C_b \neq 0.$$

Since x is not a point of C_δ or C_b, it is a point of E. Then for some $\eta>0$, $V_\eta(x) \subset E$. Then by the definition of oscillatory set,

$$X(x) \subseteq E \subset C_a,$$

which was to be proved.

A similar theorem holds when the oscillatory set of a or b is complete.

16. Simple irreducible continua. From § 11, Corollary 3, we see that each point of the first genre is a complete oscillatory set. We now turn to a consideration of limited continua, irreducible between two points and containing an everywhere dense set of points of the first genre. To save repetition in the statements of theorems, we shall call such continua *simple irreducible continua*. It may be remarked here that it is evident that the oscillatory sets of such continua about a and b are always complete. Furthermore, if x' and x'' are two points of the first genre of ab and a complete oscillatory set C is contained in the sub-continuum $x'x''$, it follows from § 12 (b) and § 11, Corollary 4, that C is a continuum of condensation of $x'x''$. The following lemmas are convenient in later work.

LEMMA I. *Let ab be a simple irreducible continuum and let C be a complete oscillatory set containing neither a nor b. Then there are sequences of points $\{x_i\}$ and $\{x_j\}$ of the first genre converging to c and lying in C_a and C_b, respectively, and $C = Dv[x_i x_j]$.*

Proof. If y_i is any point of C_a, there is a δ_i so small that $V_{\delta_i}(y_i)$ contains no point of C or C_b. But every $V_\delta(y_i)$ contains points of the first genre. Hence it is evident that any sequence $\{y_i\}$ of points of C_a converging to c can be replaced by a sequence of points of the first genre belonging to C_a and converging to c. Thus the sequences $\{x_i\}$ and $\{x_j\}$ exist.

Now for any η, $V_\eta(c)$ contains an x_i and an x_j. Hence $C \supseteq Dv\{x_i x_j\}$. On the other hand, for any given x_i and x_j there is an η so small that $V_\eta(c) \subseteq x_i x_j$. Thus $C \subseteq x_i x_j$ and hence $C \subseteq Dv[x_i x_j]$. Therefore $C = Dv[x_i x_j]$.

LEMMA II. *Let ab be a simple irreducible continuum and A be the oscillatory set about a. Then there is a sequence of points $\{x_i\}$ of the first genre converging to a and lying in A_b, and $A = Dv[a x_i]$.*

This and the corresponding lemma for the point b are proved in the same way as Lemma I.

17. THEOREM. *Let ab be a simple irreducible continuum. Then* (i) *no two complete oscillatory sets have common points*; (ii) *every incomplete oscillatory set Y contains a point whose oscillatory set is complete and contains Y*; (iii) *every point lies on one and only one complete oscillatory set*; *and* (iv) *if two oscillatory sets have common points, one is complete and contains the other, or both are contained in the same complete oscillatory set.*

Proof. (i) Let X be a complete oscillatory set and $ab = X_a + X + X_b$, where either X_a or X_b may be void. Let $Y = Y(y)$ be another complete oscillatory set. If y is a point of X_a or X_b, say the former, § 15 shows that $Y \subset X_a$; hence $X \cdot Y = 0$. If y is a point of X, then § 14 shows that

$$Y \subseteq X. \tag{1}$$

In this case we have $ab = Y_a + Y + Y_b$. If x is not a point of Y, then by § 15 X is a part of Y_a or Y_b, which is false, since $X \cdot Y \supset y$. Then x is a point of Y and hence

$$X \subseteq Y. \tag{2}$$

Then either $X \cdot Y = 0$, or by relations (1) and (2) $X = Y$.

(ii) Let Y be an oscillatory set which is not complete. If c is a point of the first genre, $Y \subset ac$ or $Y \subset cb$, by § 15. Then c is a point of Y_b or Y_a, respectively. Thus all the points of the first genre are on Y_a and Y_b. As they are everywhere dense in ab, $Y = Y \cdot \overline{Y}_a + \cdot Y\overline{Y}_b$, $Y \cdot \overline{Y}_a \neq 0$, $Y \cdot \overline{Y}_b \neq 0$. Since Y is a continuum, at least one point x of Y is a limiting point of both Y_a and Y_b. Then, by § 13, Y is a part of the oscillatory set X about x.

To show that X is complete, observe first that $X_a \subseteq Y_a$ and $X_b \subseteq Y_b$. Then, as all the points of the first genre are on X_a and X_b, X_a contains all the points of Y_a that are of the first genre. Since by § 10, $Y_a \cdot \overline{Y}_b = 0$, for every point z of Y_a any $V_\delta(z)$ contains points of the first genre lying on Y_a. Then, as x is a limiting point of Y_a, it is a limiting point of points of the first genre of Y_a and consequently a limiting point of X_a. Likewise it is a limiting point of X_b. Hence X is complete.

(iii) This follows from (i) and (ii).

(iv) If one of two oscillatory sets with common points is complete, it contains the other by (i) and (ii). If neither is complete, each lies in a complete oscillatory set by (ii). These complete oscillatory sets must be identical by (i).

18. **The aggregate of complete oscillatory sets.** Let X be any complete oscillatory set of the simple irreducible continuum ab, and let $K=\{X\}$ be the aggregate whose elements are these complete oscillatory sets. From § 17 it follows that $ab=U[X]$ and that no two sets X have common points.

The fact that $ab=X_a+X+X_b$ permits us to order the aggregate K. For, if Y is another element of K, we have seen (§ 17) that either $Y \subset X_a$ or $Y \subset X_b$. In the former case we say that Y precedes X and write $Y<X$; in the latter, that Y follows X and we write $Y>X$. It is easy to show that $X>Y$ if $Y<X$ and $X<Y$ if $Y>X$, and that, if X, Y, and Z are three complete oscillatory sets such that $X<Y$ and $Y<Z$, then $X<Z$. Thus K is a simply ordered aggregate. Furthermore K has a first and a last element, namely the complete oscillatory sets containing a and b, respectively.

We shall now proceed to show that the order type of the set K is similar to that of the aggregate whose elements are the points of a finite closed segment. Then by virtue of this similarity we shall determine a correspondence between the points of ab and those of a linear segment.

19. THEOREM. *The aggregate of complete oscillatory sets of a simple irreducible continuum ab has a dense order type.*

Proof. We must prove that, if X and Y are two complete oscillatory sets, then there is a third lying between them. Let $X<Y$. Then

$$ab = X_a + X + X_b, \tag{1}$$

and

$$ab = Y_a + Y + Y_b.$$

Since $X<Y$ it follows that

$$Y + Y_b \subseteq X_b.$$

If $Y+Y_b=X_b$, relation (1) gives

$$ab = X_a + X + Y + Y_b,$$

and

$$(X_a + X) \cdot (Y + Y_b) = 0.$$

This is an obvious contradiction, since ab is a continuum.

Hence X_b contains at least one point z not belonging to $Y+Y_b$. Therefore z is a point of Y_a and the complete oscillatory set Z containing z is a part of Y_a. Thus $Z<Y$. But, as z is also a point of X_b, $Z \subset X_b$ and $Z>X$. This proves the theorem.

20. THEOREM. *Let ab be a simple irreducible continuum. Then the aggregate K of complete oscillatory sets of ab has an open enumerable subset which is dense in itself and also dense in ab.*

Proof. Since the set of points of the first genre is dense in ab, there is an enumerable subset of this which is also dense in ab; call it E. Also, if a or b is of the first genre, we can assume that E contains neither of these.

Now let X and Y be two elements of K and let $X<Y$. By § 19 there is an element Z of K such that

$$X < Z < Y.$$

Then Z contains no points of the continua X_a+X and $Y+Y_b$. Then if z is a point of Z, for δ sufficiently small

$$V_\delta(z) \subset X_b \quad \text{and} \quad V_\delta(z) \subset Y_a.$$

But E is dense in ab; hence $V_\delta(z)$ contains a point e of E. As $\tau(e)=0$, the complete oscillatory set containing e is e itself. Then the complete oscillatory set of e is a part of both X_b and Y_a. Hence

$$X < e < Y.$$

Therefore E considered as an aggregate of complete oscillatory sets is dense in K and obviously dense in itself. It is also evident that it has no first and no last element.

21. THEOREM. *Let ab be a simple irreducible continuum, and let K be the aggregate of complete oscillatory sets of ab. Then the order type of K is that of a finite segment.*

Proof. Let $K=P+Q$ be a partition such that every element of P precedes every element of Q. Let X be any element of P and let Y be any element of Q; then $P=\{X\}$ and $Q=\{Y\}$. Also $B=U[X]$ and $C=U[Y]$. Then we have

$$ab = B + C.$$

Suppose now that P has no last and Q no first element. Let Y be *any* element of Q. Then there is an element Y' of Q preceding Y and every element X of P precedes Y'. Then $B=U[X] \subset Y_a'$ and

$$\bar{B} \subseteq Y_a' + Y' \subset Y_a.$$

Hence $\bar{B}\cdot Y=0$ and no point of $\bar{B}$ lies in $C=U[Y]$.

In the same way we show that $B\cdot\bar{C}=0$. Since $\bar{B}\cdot C+B\cdot\bar{C}=0$, we have a contradiction, as ab is a continuum. Hence either P has a last, or Q has a

first element. As it has been proved in § 19 that K is dense, this shows that K is continuous.*

If we omit from K the complete oscillatory sets of a and b, the resulting set K' is also continuous and by § 20 K' contains an open enumerable aggregate dense in itself and dense in K'. Therefore the order type of K' is that of the linear continuum† and K is similar to a finite closed segment.

22. The effect of the theorem just proved is to set up a uniform correspondence between the points of a finite segment $(0 \leqq t \leqq 1)$ and the complete oscillatory sets of a simple irreducible continuum $ab = \{x\}$. This defines a correspondence between the points of ab and those of the segment, which need not be uniform. If we denote this by $x = f(t)$, we have a function which is in general multivalued. It is, moreover, but a slight extension of the ordinary notion of continuity if we say that $f(t)$ is continuous at a point t' when $f(t')$ is a single point x' and for every $\epsilon > 0$, the aggregate of images of all points t in some $V_\delta(t')$ lie in $V_\epsilon(x')$.

THEOREM. *Let $ab = \{x\}$ be a simple irreducible continuum. Then there is a correspondence $x = f(t)$ between the points of ab and those of the unit segment $T = (0 \leqq t \leqq 1)$ having the following properties:*

(i) *To each point t corresponds one and only one point of the first genre or one and only one complete oscillatory set.*

(ii) *$f(t)$ is continuous at each point t whose image is a single point.*

(iii) *At a point t_0 whose image is not a single point the corresponding oscillatory set $X_0 = \{x_0\}$, $x_0 = f(t_0)$, is the set of accumulation‡ of the sets $x = f(t)$ as $t \to t_0$.*

(iv) *The images of 0 and 1 are the complete oscillatory sets of a and b, respectively.*

Proof. The correspondence in question is that defined by the similarity of the aggregate of complete oscillatory sets $K = \{X\}$ of ab to any finite closed segment. This gives us (i) and (iv) at once. It remains to prove (ii) and (iii).

Let t_0 be a point whose image is the single point x_0. Then $\tau(x_0) = 0$ and this fact, with § 16, shows that there are two points of the first genre, x' and x'', such that $x' < x_0 < x''$ and the irreducible sub-continuum $x'x''$ is a

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* See F. Hausdorff, *Grundzüge der Mengenlehre*, p. 90.

† Ibid., p. 101.

‡ This notion is defined by Janiszewski (loc cit., p. 93) for the case of an enumerable system of sets. The following is a natural extension. Let $\{f(t)\}$ denote a system of sets depending on the parameter t, which ranges over an interval and let t_0 be any point in this interval. Let F be the class of all points $\{x\}$ such that, for every $\epsilon > 0$, $S_\epsilon(x)$ contains a point of some $f(t)$ for at least one t in every $V_\delta(t_0)$, $\delta > 0$. Then F is the aggregate of accumulation of the sets $f(t)$ as $t \to t_0$.

part of $V_\epsilon(x_0)$ for a fixed positive ϵ. Let $x'=f(t')$ and $x''=f(t'')$. Then by similarity, $t'<t_0<t''$. Now for δ sufficiently small $V_\delta(t_0) \subset t't''$. Hence the images of points in $V_\delta(t_0)$ lie in $x'x''$, since $x'<X<x''$, when $t'<t<t''$. But then they lie in $V_\epsilon(x_0)$ and we have continuity. The cases where $t_0=0$ or $t_0=1$ are treated similarly.

Now let t_0 be a point for which the corresponding oscillatory set X_0 contains more than one point. Let x_0 be a point of X_0 which is a common limiting point of the saturated semi-continua X_a and X_b of $ab-X_0$. Then, by § 16, X_a and X_b contain respectively sequences $\{x_i\}$ and $\{x_j\}$ of points of the first genre converging to x_0 and $X_0=Dv[x_i,x_j]$. Hence for any $\epsilon>0$ there is an x' and an x'', points of the first genre, such that

$$x' < X_0 < x'' \quad \text{and} \quad X_0 \subset x'x'' \subset V_\epsilon(X_0).$$

Let $x'=f(t')$, $x''=f(t'')$. Then $t'<t_0<t''$. For δ sufficiently small, then, the images of all points in $V_\delta(t_0)$ will lie in $x'x''$ and consequently in $V_\epsilon(X_0)$. Hence, as $t\to t_0$, the points of accumulation of the corresponding points x will be contained in X_0.

Now let x be *any* point in X_0. Let $\epsilon>0$. For any $\delta>0$ there are points t_i and t_j in $V_\delta(t_0)$ such that $t_i<t_0<t_j$ and the images of t_i and t_j are points of the first genre. Let $x_i=f(t_i)$ and $x_j=f(t_j)$. By § 16, X_0 is a continuum of condensation of x_ix_j; hence $V_\epsilon(x)$ contains at least one point x' of x_ix_j which is not on X_0. But if $x'=f(t')$, then t' lies in $V_\delta(t_0)$. Thus each point of X_0 is a point of accumulation of the points $x=f(t)$ as $t\to t_0$, by definition.

This, with the preceding paragraph, completes the proof of (iii).

23. The nature of the correspondence involved in the theorem just proved is made clearer by relating it to the function theory. If $x=f(t)$ is a limited one-valued function defined for a set E of values of t everywhere dense in the interval $a\leqq t\leqq b$ and continuous at each point, it is well known that in general there is no function continuous at every point of ab and equal to $f(t)$ at the points of E, for the reason that, at a point c not in E, $\lim_{t\to c} f(t)$ may not exist. If, however, we agree to set $F(t)=f(t)$ at the points of E and to assign to $F(t)$ at $t=c$ the aggregate of limiting values of $f(t)$ as $t\to c$ over all possible sequences in E and for each point c this aggregate is a continuum, we have a multivalued function possessing many of the attributes of continuity. The graph is easily seen to be a continuum and, for any point c, all of the limits of $F(t)$ as $t\to c$ are values of $F(c)$. Such a function is $x=\sin(1/t)$ for $t\neq 0$, $-1\leqq x\leqq 1$ for $t=0$. If $F(t)$ happens to be one-valued in ab, it is an ordinary continuous function; if $G(t)=F(t)$ at points where the latter is one-valued and at other points $G(t)$ has one of the values of $F(t)$, then $G(t)$ is a one-valued pointwise discontinuous function.

By the method of condensation of singularities* it is easy to build up simple irreducible continua which have an everywhere dense set of points of the second genre. The previous theorem therefore applies, although such continua have only one prime part.

We now turn to the converse theorem.

24. THEOREM. *Let $x=f(t)$ denote a correspondence of the set $C=\{x\}$ to the closed segment $T=\{t\}$ having the following properties:*

(i) *To each t corresponds a point or a point set forming a limited continuum.*

(ii) *No point of C corresponds to more than one point of T.*

(iii) *$f(t)$ is one-valued and continuous at a set of points everywhere dense in T.*

(iv) *At points t' where $f(t)$ is many-valued, $f(t')$ is the aggregate of accumulation of $f(t)$ as $t\to t'$.*

Then C is a simple irreducible continuum.

Proof. That C is limited follows from the definition of the correspondence. It is also easy to see that the image in C of a closed set in T is closed, and vice versa.

It will now be shown that the image in C of a sub-continuum of T is a continuum, and vice versa. Let F be a sub-continuum of T and let G be its image in C. If G is not a continuum, there is a partition $G=G_1+G_2$, where both sets are closed and $G_1\cdot G_2=0$. Suppose that F_1 and F_2 are the images in T of G_1 and G_2 respectively. F_1 and F_2 are closed. They have no common point t, for the image of t is a continuum or a point and hence must lie wholly in G_1 or G_2. Thus $F_1\cdot F_2=0$, which contradicts the hypothesis that F is a continuum. Hence G is a continuum.

Conversely, let G be a sub-continuum of C and let F be the subset of T to which correspond† points in G. The set F is closed. If F is not a continuum, we have $F=F_1+F_2$, $F_1\cdot F_2=0$, and F_1 and F_2 are closed. Then $G=G_1+G_2$, where G_1 and G_2 are closed and $G_1\cdot G_2=0$, contrary to the hypothesis. Hence, if G is a continuum, so is F.

It will now be shown that, if the image of a point t_0 of T is a single point x_0, then x_0 is a point of the first genre. For every $\epsilon>0$ there is a $\delta>0$ such that x is a point of $V_\epsilon(x_0)$ for every t in $V_\delta(t_0)$. Let $t'<t_0$ and $t''>t_0$ be points in $V_\delta(t_0)$ whose images are single points. Then the image in C of $t't''$ is a continuum contained wholly in $V_\epsilon(x_0)$ while the images of the intervals

* See E. W. Hobson, *Theory of Functions of a Real Variable*, p. 618.

† It should be noted that in general G will be only a part of the set of points of C corresponding to points of F.

0, t' and t'', 1 are continua neither of which contains x_0. Hence for $\eta>0$ sufficiently small, $V_\eta(x_0)$ is contained in the image of $t't''$. Thus for any $\epsilon>0$ there is an $\eta>0$ such that there is a sub-continuum of C lying in $V_\epsilon(x_0)$ and containing $V_\eta(x_0)$. Therefore $\tau(x_0)=0$.

Since the points of C are either unique images of points of T or are limiting points of this set, this shows that the points of the first genre are everywhere dense in C.

Now let $a=f(0)$ and $b=f(1)$. If D is a sub-continuum of C which contains a and b, its image in T must be T itself since it is a continuum containing the points 0 and 1. Hence D must contain all the points of the first genre of C. As these are everywhere dense in C, $D=C$. Hence C is irreducible between a and b.

This completes the proof of the theorem.

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APPLICATION OF THE THEORY OF RELATIVE CYCLIC FIELDS TO BOTH CASES OF FERMAT'S LAST THEOREM

BY

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If, for p an odd prime,

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers of x, y, and z prime to each other, $z \not\equiv 0 \pmod{p}$, then in another paper† I gave the relation

$$\prod_{r=1}^{k-1} \prod_{r=1}^{[rp/k]} (x + \alpha^{[1:r]}y) = \alpha^{-kyq(k)/(x+y)}\omega^p, \tag{2}$$

where k is an integer, $1 < k < p$;

$$q(k) = \frac{k^{p-1} - 1}{p};$$

$[s]$ is the greatest integer in s; ω is an integer in the field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$; $[1 : r]$ is the integer i in the relation $ri \equiv 1 \pmod{p}$, and if a fraction f/g occurs as an exponent of α, then that exponent is the integer u in the relation $f \equiv gu \pmod{p}$.

In the present paper I shall develop a new line of attack on the Last Theorem by the introduction of power characters in the field $\Omega(e^{2i\pi/ph})$, h prime to p, in connection with (2).

1. Let n be a prime $\not\equiv 0$ or $1 \pmod{p}$ and suppose that $xyz \not\equiv 0 \pmod{n}$; then

$$x^{n-1} - y^{n-1} \equiv 0 \pmod{n}. \tag{3}$$

If β is a primitive $(n-1)$th root of unity then in the field $\Omega(\beta)$ we have

$$(n) = \mathfrak{q}_1\mathfrak{q}_2 \cdots \mathfrak{q}_{\varphi(n-1)}$$

where the $\mathfrak{q}$'s are distinct prime ideals, and $\varphi(n-1)$ is the indicator of $n-1$. We may take as one of the $\mathfrak{q}$'s the ideal

$$\mathfrak{q} = (\beta - r, n)$$

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† Annals of Mathematics, ser. 2, vol. 21 (1919), p. 78.

where r is a primitive root of n. Then (3) gives

$$\prod_{s=0}^{n-2}(x+\beta^s y)\equiv 0 \pmod{\mathfrak{q}};$$

hence there is an integer a in the set $1, 2, \cdots, n-2$, such that

$$x+\beta^a y\equiv 0 \pmod{\mathfrak{q}} \tag{4}$$

if we note that $x+y\not\equiv 0 \pmod{\mathfrak{q}}$ since $z\not\equiv 0 \pmod{n}$. Now in the field $\Omega(\alpha\beta)$ we have, if θ is any integer such that (θ) is prime to (p) and the ideal prime $\mathfrak{p}$, with $\mathfrak{p}$ also prime to (p), if $c=N(\mathfrak{p})-1$,

$$\theta^c\equiv 1 \pmod{\mathfrak{p}},$$

$N(\mathfrak{p})$ being the norm of $\mathfrak{p}$, by Fermat's generalized theorem, and consequently there is an integer s such that

$$\theta^{c/p}\equiv\alpha^s \pmod{\mathfrak{p}}$$

since $N(\mathfrak{p})\equiv 1 \pmod{p}$. Set

$$\left\{\frac{\theta}{\mathfrak{p}}\right\}=\alpha^s.$$

It follows that θ is congruent to the pth power of an integer in $\Omega(\alpha\beta)$ if and only if

$$\left\{\frac{\theta}{\mathfrak{p}}\right\}=1.$$

If the ideal $\mathfrak{P}=\mathfrak{p}_1'\mathfrak{p}_2'\cdots\mathfrak{p}_c'$ then we use as definition

$$\left\{\frac{\theta}{\mathfrak{P}}\right\}=\left\{\frac{\theta}{\mathfrak{p}_1'}\right\}\left\{\frac{\theta}{\mathfrak{p}_2'}\right\}\cdots\left\{\frac{\theta}{\mathfrak{p}_c'}\right\},$$

the $\mathfrak{p}$'s being prime ideals in $\Omega(\alpha\beta)$. It follows from the definition that if ψ is an integer in the field $\Omega(\beta)$, then since $n-1\not\equiv 0 \pmod{p}$,

$$\left\{\frac{\psi}{\mathfrak{Q}}\right\}=1, \tag{4a}$$

$\mathfrak{Q}$ being an ideal in $\Omega(\beta)$, and if ζ is an integer in $\Omega(\alpha\beta)$ and ζ_i denotes the integer obtained by the substitution (α/α^i), i prime to p, then

$$\left\{\frac{\zeta_i}{\mathfrak{Q}}\right\}=\left\{\frac{\zeta}{\mathfrak{Q}}\right\}^i. \tag{4b}$$

Let

$$\mathfrak{q}=\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_d,$$

the $\mathfrak{p}$'s being prime ideals in $\Omega(\alpha\beta)$.

We shall now show that

$$\left\{\frac{\alpha}{\mathfrak{q}}\right\} = \alpha^{q(n)}. \tag{4c}$$

Let

$$\begin{aligned} N(\mathfrak{p}_1) &= 1 + w_1 p, \\ N(\mathfrak{p}_2) &= 1 + w_2 p, \\ &\cdots \quad \cdots \\ N(\mathfrak{p}_d) &= 1 + w_d p; \end{aligned}$$

multiplication gives

$$N(\mathfrak{q}) \equiv 1 + p \sum w \pmod{p^2}, \qquad \frac{N(\mathfrak{q}) - 1}{p} \equiv \sum w \pmod{p}.$$

But $w_s = (N(\mathfrak{p}_s) - 1)/p$, so that

$$\frac{N(\mathfrak{q}) - 1}{p} \equiv \sum_{s=1}^{d} \frac{N(\mathfrak{p}_s) - 1}{p} \pmod{p},$$

and (4c) follows immediately from

$$\left\{\frac{\alpha}{\mathfrak{q}}\right\} = \prod_{s=1}^{d} \left\{\frac{\alpha}{\mathfrak{p}_s}\right\},$$

since

$$\left\{\frac{\alpha}{\mathfrak{q}}\right\} = \alpha^{e/p};$$

and

$$N(\mathfrak{q}) = n^{p-1}.$$

Now take power characters[1] of each member of (2) with respect to $\mathfrak{q}$, and since $\mathfrak{q}$ is prime to (p) and (z) and therefore to $(x+\alpha^e y)$, we have

$$\prod_{r=1}^{k-1} \prod_{r=1}^{[rp/k]} \left\{\frac{x + \alpha^{[1:r]} y}{\mathfrak{q}}\right\} = \left\{\frac{\alpha}{\mathfrak{q}}\right\}^{-kyq(k)/(x+y)}. \tag{5}$$

Now also by (4)

$$\{(x + \alpha^e y)/\mathfrak{q}\} = \{(x + \beta^a y + y(\alpha^e - \beta^a))/\mathfrak{q}\} = \{y/\mathfrak{q}\}\{(\alpha^e - \beta^a)/\mathfrak{q}\}.$$

By (4a)

$$\left\{\frac{y}{\mathfrak{q}}\right\} = 1,$$

so that

$$\left\{\frac{x+\alpha^s y}{\mathfrak{q}}\right\} = \left\{\frac{\alpha^s-\beta^a}{\mathfrak{q}}\right\}.$$

We also have by (4b)

$$\left\{\frac{\alpha^a-\beta^a}{\mathfrak{q}}\right\} = \left\{\frac{\alpha-\beta^a}{\mathfrak{q}}\right\}^s.$$

Applying these relations to (5) we obtain with (4c)

$$\left\{\frac{\alpha-\beta^a}{\mathfrak{q}}\right\}^{\Sigma[1:r]} = \alpha^{-kyq(k)q(n)/(x+y)},$$

and since*

$$-kq(k) \equiv \sum [1:r] \pmod{p},$$

we have

$$\left\{\frac{\alpha-\beta^a}{\mathfrak{q}}\right\}^{-kq(k)} = \alpha^{-kyq(k)q(n)/(x+y)}.$$

For $k=p-1$ we have $q(k)\not\equiv 0 \pmod{p}$ so that

$$\left\{\frac{\alpha-\beta^a}{\mathfrak{q}}\right\} = \alpha^{yq(n)/(x+y)},$$

or since

$$\left\{\frac{\beta^{-a}}{\mathfrak{q}}\right\} = 1,$$

then

$$\left\{\frac{\alpha\beta^{-a}-1}{\mathfrak{q}}\right\} = \alpha^{yq(n)/(x+y)}. \tag{6}$$

Note that $(\alpha\beta^{-a}-1)$ is a unit in $\Omega(\alpha\beta)$.

If we write

$$\left\{\frac{\alpha\beta^{-a}-1}{\mathfrak{q}}\right\} = \alpha^i$$

and $i=\text{ind}\,(\alpha\beta^{-a}-1)$, then (6) shows that for some value of a included in the set $1, 2, \cdots, n-2$,

$$\text{ind}(\alpha\beta^a-1) - \frac{yq(n)}{x+y} \equiv 0 \pmod{p}. \tag{7}$$

* Vandiver, loc. cit., p. 77, relations 17.

This is equivalent to the relation

$$\prod_{a=1}^{n-2} (\text{. nd } (\alpha\beta^a - 1) - \frac{y}{x+y} q(n)) \equiv 0 \pmod{p}. \tag{7a}$$

2. Let us now consider the first case of Fermat's Last Theorem; that is, when $xyz \not\equiv 0 \pmod{p}$. Let $-x/y = t$; then it follows from (1) that the relation

$$\prod_{a=1}^{n-2} ((1-v) \text{ ind } (\alpha\beta^a - 1) - q(n)) \equiv 0 \pmod{p} \tag{8}$$

holds if v has any of the six values

$$t, \quad 1-t, \quad \frac{1}{t}, \quad \frac{1}{1-t}, \quad \frac{t}{t-1}, \quad \frac{t-1}{t}. \tag{9}$$

This criterion for (1) when $xyz \not\equiv 0 \pmod{p}$ was obtained under the assumption that xyz was prime to n. If either x, y or z is divisible by n, then it follows by Furtwängler's theorem* that $q(n) \equiv 0 \pmod{p}$. We may then state

THEOREM I. *If $x^p + y^p + z^p = 0$ is satisfied in integers none zero and all prime to the odd prime p, v is any number in the set (9), then for $\alpha = e^{2i\pi/p}$, $\beta = e^{2i\pi/(n-1)}$*

$$q(n) \prod_{a=1}^{n-2} ((1-v) \text{ ind } (\alpha\beta^a - 1) - q(n)) \equiv 0 \pmod{p},$$

where $\mathfrak{q} = (\beta - r, n)$, r is a primitive root of n,

$$\left\{\frac{\alpha\beta^a - 1}{\mathfrak{q}}\right\} = \alpha^i, \quad q(n) = \frac{n^{p-1} - 1}{p},$$

$i = \text{ind } (\alpha\beta^a - 1)$, and n is a prime $\not\equiv 0$ or $1 \pmod{p}$.

The relation (7) is equivalent to

$$(1-t) \text{ ind } (\alpha\beta^a - 1) - q(n) \equiv 0 \pmod{p}. \tag{10}$$

Because of (9), there is also an integer b in the set $1, 2, \cdots, n-2$ such that

$$t \text{ ind } (\alpha\beta^b - 1) - q(n) \equiv 0 \pmod{p}. \tag{11}$$

Eliminating t from (10) and (11) gives

$$\text{ind } (\alpha\beta^a - 1) \text{ ind } (\alpha\beta^b - 1) - q(n)(\text{ ind } (\alpha\beta^a - 1) + \text{ ind } (\alpha\beta^b - 1)) \equiv 0 \pmod{p}.$$

This gives

* Wiener Berichte, IIa, 1912, 589–92.

THEOREM II. *If $x^p+y^p+z^p=0$ is satisfied in integers none zero and all prime to the odd prime p, then*

$$q(n)\prod_{a,b}(\operatorname{ind}(\alpha\beta^a-1)\operatorname{ind}(\alpha\beta^b-1) - q(n)(\operatorname{ind}(\alpha\beta^a-1)+\operatorname{ind}(\alpha\beta^b-1))\equiv 0 \pmod p),$$

where a and b each range independently over the integers $1, 2, \cdots, n-2$, the other symbols being defined as in Theorem I.

It will be noted that these criteria are independent of x, y and z.

For $n=3$, $\mathfrak{q}=(3)$, and

$$\left\{\frac{\alpha\beta-1}{3}\right\}=\left\{\frac{-\alpha-1}{3}\right\}=\left\{\frac{\alpha+1}{3}\right\}=\left\{\frac{\alpha^{\frac{1}{2}}}{3}\right\}\left\{\frac{\alpha^{\frac{1}{2}}+\alpha^{-\frac{1}{2}}}{3}\right\}=\left\{\frac{\alpha^{\frac{1}{2}}}{3}\right\}=\alpha^{q(3)/2}.$$

Using this in connection with the criteria of Theorem II, we have

$$q(3)\left(\frac{1}{4}(q(3))^2-2\cdot\frac{1}{2}(q(3))^2\right)\equiv 0 \pmod p,$$

whence $q(3)\equiv 0 \pmod p$ assuming $p>3$.

Take $n=5$; then $n-1=4$ and $\Omega(\beta)$ is the field $\Omega(i)$ and we may set $\mathfrak{q}=(2-i)$. We have

$$(x-y)(x^2+y^2)\equiv 0 \pmod 5$$

and similarly for (x, z), (z, y) in lieu of (x, y). It then follows easily that one of the integers $x-y$, $x-z$, $y-z$ is divisible by 5. If $x-y\equiv 0 \pmod 5$ it follows from (7) that $q(5)\equiv 0 \pmod p$ unless $x-y\equiv 0 \pmod p$. This is equivalent to the condition that the set (9) satisfies

$$q(5)(t+1)(t-2)(t-\tfrac{1}{2})\equiv 0 \pmod p. \tag{12}$$

Theorem I also gives

$$q(5)\prod_{a=1}^{3}((1-t)\operatorname{ind}(\alpha\beta^a-1)-q(5))\equiv 0 \pmod p. \tag{13}$$

As in the case $n=3$ we find

$$\left\{\frac{\alpha\beta^2-1}{\mathfrak{q}}\right\}=\alpha^{q(5)/2}.$$

Hence if we write

$$\operatorname{ind}(\alpha\beta^a-1)=I_a$$

we have from (13)

$$q(5)(t+1)((1-t)I_1-q(5))((1-t)I_3-q(5))\equiv 0 \pmod p. \tag{14}$$

Now also

$$I_1 + I_3 = \text{ind}\,(\alpha^2 + 1) \equiv q(5) \pmod{p},$$

so that

$$I_3 \equiv q(5) - I_1 \pmod{p}. \tag{14a}$$

Comparing (12) and (14) it follows that

$$q(5)\big((1-t)I_1 - q(5)\big)\big((1-t)I_3 - q(5)\big) \equiv 0 \pmod{p}$$

for $t=2$ and $t=\frac{1}{2}$, and these values give in each case, using (14a),

$$q(5)\big(I_1 + q(5)\big)\big((I_1 - 2q(5)\big) \equiv 0 \pmod{p}.$$

3. We shall now consider the second case of the Last Theorem. In (7a) assume $y\equiv 0 \pmod{p}$; then we obtain

$$\prod_{\alpha=1}^{n-2} \text{ind}\,(\alpha\beta^\alpha - 1) \equiv 0 \pmod{p},$$

under the assumption that x, y and z are each prime to n. If x or z is divisible by n then $q(n)\equiv 0 \pmod{p}$, but this does not necessarily hold when $y\equiv 0 \pmod{n}$. Hence

THEOREM III. *If p is an odd prime and $x^p+y^p+z^p=0$ is satisfied in integers, none zero, $y\equiv 0 \pmod{p}$, with $xz\not\equiv 0 \pmod{p}$, then either $y\equiv 0 \pmod{n}$ or*

$$q(n) \prod_{\alpha=1}^{n-2} \text{ind}\,(\alpha\beta^\alpha - 1) \equiv 0 \pmod{p},$$

the symbols being defined as in Theorem I.

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